Turing Meets Shannon: Computable Sampling Type Reconstruction With Error Control

Holger Boche, *Fellow, IEEE,* and Ullrich J. Mönich, *Senior Member, IEEE*

***Abstract*—The conversion of analog signals into digital signals and vice versa, performed by sampling and interpolation, respectively, is an essential operation in signal processing. When digital computers are used to compute the analog signals, it is important to effectively control the approximation error. In this paper we analyze the computability, i.e., the effective approximation of bandlimited signals in the Bernstein spaces $\mathcal{B}_\pi^p$, $1 \leq p < \infty$, and of the corresponding discrete-time signals that are obtained by sampling. We show that for $1 < p < \infty$, computability of the discrete-time signal implies computability of the continuous-time signal. For $p = 1$ this correspondence no longer holds. Further, we give a necessary and sufficient condition for computability and show that the Shannon sampling series provides a canonical approximation algorithm for $p > 1$. We discuss BIBO stable LTI systems and the time-domain concentration behavior of bandlimited signals as applications.**

***Index Terms*—Sampling, discrete-time signal, continuous-time signal, effective approximation, approximation error**

I. Introduction

According to Shannon's sampling theorem, a bandlimited signal with finite energy is uniquely determined by its samples taken at the Nyquist rate, and the continuous-time signal can be recovered from the samples by means of the Shannon sampling series. This fact allows us to connect the continuous-time and discrete-time worlds by sampling and interpolation. Interestingly, when applying sampling or interpolation, many properties and characteristics of the signal carry over from one domain into the other. For example, the energy of a signal can be determined from the continuous-time signal or the discrete-time signal and is the same in both domains.

The sampling theorem has a long history and many famous names, such as Whittaker [2], Ogura [3], Kotel'nikov [4], Raabe [5], and Shannon [6] are linked with its discovery. For an historical treatment of the sampling theorem, see [7]–[9]. Shannon originally employed the sampling theorem to bridge the analog and digital worlds, which enabled him to study the communication capacity of continuous channels [6]. Today, the sampling theorem is of fundamental importance in communications [10], [11]. For example, it is used in communication systems to convert the digital baseband signal into the actual analog waveform that is transmitted over the wireless channel. Shannon introduced the sampling theorem for bandlimited signals with finite energy. By now, many authors have extended this result in different directions, e.g., to sampling theorems for more general signal spaces [12], [13], non-bandlimited signals [14], and stochastic processes [15]. Other extensions deal with non-equidistant sampling [16], missing samples [17], multiband sampling [18], multidimensional sampling [19], and sampling in the context of lattice functions [20].

In this paper we consider the Bernstein spaces $\mathcal{B}_\pi^p$, i.e., bandlimited signals with finite L^p-norm as characteristic time-domain behavior. In general, such signals cannot be represented in closed form, e.g., in optimization tasks or filter design problems. Hence the approximation of such signals and the control of the approximation error is important. The approximation and the error control, can both be done in either the continuous-time or the discrete-time domain. The question is how the control of the approximation error in one domain translates into a control of the error in the other domain. We will show that for $1 < p < \infty$ we have a coupling of the approximation errors, whereas for $p = 1$ such a coupling does not exist. This failure for $p = 1$ is noteworthy because the space $\mathcal{B}_\pi^1$ is practically relevant, e.g., for modeling the impulse responses of bounded-input bounded-output (BIBO) stable linear time-invariant (LTI) systems.

Nowadays, signal processing is often done on digital hardware, such as microprocessors, field programmable gate arrays (FPGAs), or digital signal processors (DSPs), and hence questions of computability arise. In order to study the question of computability, we employ the concept of Turing computability.

A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [21]–[24]. Although the concept is very simple, a Turing machine is capable of simulating any given algorithm. Turing machines have no limitations in terms of memory or computing time, and hence provide a theoretical model that describes the fundamental limits of any practically realizable digital computer. Computability is a mature topic in computer science [21]–[26]. In the signal processing literature, however, this aspect has not received much attention so far.

Computability is important for the control of the approximation error if digital hardware is used to compute the signals. One of the key concepts of computability is the effective, i.e., algorithmic control of the approximation error. If a signal is

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Holger Boche is with the Technische Universität München, Lehrstuhl für Theoretische Informationstechnik, Germany, and the Munich Center for Quantum Science and Technology (MCQST), Schellingstr. 4, Munich, Germany. e-mail: boche@tum.de. Ullrich J. Mönich is with the Technische Universität München, Lehrstuhl für Theoretische Informationstechnik, Germany. e-mail: moenich@tum.de.

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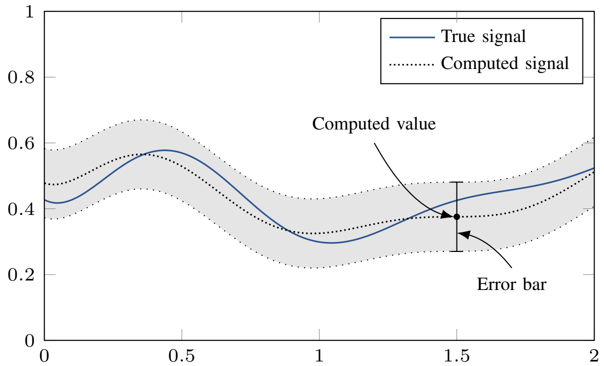

Fig. 1. For a computable signal we can always determine an error bar and can then be sure that the true value lies within the specified error range.

computable, then for every prescribed error tolerance ϵ we can compute an approximation that is ϵ-close to the desired signal. This is illustrated in Fig. 1. In contrast to classical approximation theory, where the mere mathematical existence of an approximation is sufficient, the essential point for computability is that the approximation can be algorithmically computed in a finite number of steps. The exact definitions of effective convergence and a computability for signals will be given in Section II.

To the best of our knowledge, the approximation of bandlimited signals—although being a classical topic in signal processing [14], [27]–[32]—has never been studied from a computational point of view. It is often assumed that in principle, it is possible to approximate signals and systems arbitrarily well, for example by increasing the sampling rate and by using a finer quantization. That this not necessarily the case has recently been demonstrated for the Fourier transform, the bandlimited interpolation, and the Wiener filter [33]–[37]. In [33], [34] the Fourier transform has been studied with respect to computability, in [35] the Fourier series, and in [37], [38], the spectral factorization.

As we will see in this paper, the transition from the discrete-time domain into the continuous-time domain can be problematic. A typical signal processing problem is as follows: We have a discrete-time signal, which represents the samples of a bandlimited continuous-time signal, and we want to reconstruct or approximate this continuous-time signal from the samples. In our previously mentioned communication scenario, the discrete-time signal is the digital baseband signal that has to be converted into an analog waveform which can be transmitted. We will show in Section V that even if the discrete-time signal is a well-behaved computable sequence, the corresponding bandlimited continuous-time signal is not necessarily computable. In this case, the error that is made in the approximation of the continuous-time signal cannot be algorithmically controlled.

This result has further implications, for example for the approximation of BIBO stable LTI systems, and the algorithmic characterization of the time-domain concentration behavior of bandlimited signals. In Section VII-A we will see that there exist BIBO stable LTI systems with bandlimited and absolutely integrable impulse responses for which the discrete-time BIBO norm is computable, but where the continuous-time BIBO-norm cannot be algorithmically determined on a digital computer. As a further example, we discuss in Section VII-B problems related to the algorithmic characterization of the time-domain concentration behavior of bandlimited signals. In both examples we will see the importance of an algorithmic control of the approximation error, which can only be guaranteed if the involved signals are computable.

We present the basic definitions next. By c_0 we denote the set of all sequences that vanish at infinity, and by $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, we denote the usual spaces of pth-power summable sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with the norm $\|x\|_{\ell^p} = (\sum_{k=-\infty}^{\infty}|x(k)|^p)^{1/p}$. For $\Omega \subset \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|f\|_p = \left(\int_\Omega |f(t)|^p \, \mathrm{d}t\right)^{1/p}$ and $L^\infty(\Omega)$ the space of all measurable functions for which the essential supremum norm $\|f\|_\infty = \operatorname{ess\,sup}_{t\in\Omega}|f(t)|$ is finite. By $\hat{f}$ we denote the Fourier transform of a function f, and by $f|_\mathbb{Z}$ the sequence $\{f(k)\}_{k\in\mathbb{Z}}$, which is the restriction of f to the set $\mathbb{Z}$. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \leq p \leq \infty$, consists of all entire functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [32, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line, i.e., $\|f\|_{\mathcal{B}_\sigma^p} = \|f\|_p$. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. $\mathcal{B}_\sigma^2$ is the frequently used space of bandlimited functions with bandwidth σ and finite energy, and $\mathcal{B}_\sigma^\infty$ the space of all bandlimited functions that are bounded on the real axis. $\mathcal{B}_{\sigma,0}^\infty$ denotes the space of all functions in $\mathcal{B}_\sigma^\infty$ that vanish on the real axis at infinity.

In addition to $\mathcal{B}_\sigma^2$, i.e., the space of all bandlimited functions with finite energy, the spaces $\mathcal{B}_\sigma^1$ and $\mathcal{B}_{\sigma,0}^\infty$ are of particular importance in practical applications. For example, functions in $\mathcal{B}_\sigma^1$ are used to describe the impulse response of BIBO stable LTI systems operating on bandlimited signals, whereas the input and output signals are signals in $\mathcal{B}_{\sigma,0}^\infty$. We will discuss this in more detail in Section VII-A. The space $\mathcal{B}_{\sigma,0}^\infty$ is also used to model the peak-to-average power ratio (PAPR) problem in communication systems that employ orthogonal frequency division multiplexing (OFDM) to [39].

The structure of the manuscript is as follows. In Section II we introduce the concepts of computability. Then, in Section III we further discuss the relevance of the problem and give a necessary and sufficient condition for the computability of the continuous-time signal. The Shannon sampling series as a canonical approximation algorithm is studied in Section IV. In Section V we provide a refined analysis and highlight the differences between $\mathcal{B}_\pi^p$, $1 < p < \infty$, and $\mathcal{B}_\pi^1$. The computability of functions in $\mathcal{B}_\pi^2$ is treated in Section VI. Before concluding the manuscript with Section VIII, in Section VII we discuss as applications BIBO stable LTI systems and the time-domain concentration behavior of signals.

II. Computability

The theory of computability is a well-established field in computer sciences [21]–[26]. Alan Turing introduced the concept of a computable real number in [21], [22]. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and $r_n = (-1)^{s(n)} a(n)/b(n)$, $n \in \mathbb{N}$. A recursive function is a function mapping natural numbers into natural numbers that is computable by a Turing machine. For a precise definition of a recursive function, see [40]. For the purposes in this paper, the exact definition is not of importance, it only matters that recursive functions are exactly those functions that are computable by a Turing machine.

A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$, we have $|x - r_n| \leq 2^{-M}$ for all $n \geq \xi(M)$. This form of convergence with a computable control of the approximation error is called effective convergence. Note that if a computable sequence of real numbers $\{x_n\}_{n\in\mathbb{N}}$ converges effectively to a limit x, then x is a computable real number [25, p. 20, Corollary 2a]. A non-computable real number was, for example, constructed in [41]. By $\mathbb{R}_c$ we denote the set of computable real numbers and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers.

A set $A \subset \mathbb{N}$ is called recursively enumerable if $A = \emptyset$ or A is the range of a recursive function. A set $A \subset \mathbb{N}$ is called recursive if both A and $\mathbb{N} \setminus A$ are recursively enumerable. We say that a set $A \subset \mathbb{N}$ is a recursively enumerable non-recursive set if A is recursively enumerable but not recursive, i.e., if A is recursively enumerable but $\mathbb{N}\setminus A$ is not recursively enumerable. Such recursively enumerable non-recursive sets exist [40, 4.4 Proposition, p. 19] and will be of great importance for the results in this paper. For every recursively enumerable non-recursive set $A \subset \mathbb{N}$ there exists a recursive enumeration of A, i.e., a recursive function $\phi_A\colon \mathbb{N} \to A$ that is surjective and injective. In this paper, the series

$$\sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}},$$

where $A \subset \mathbb{N}$ is a recursively enumerable non-recursive set, will play an important role. We will discuss the relevant properties next. First, we note that

$$s_M = \sum_{n=1}^{M} \frac{1}{2^{\phi_A(n)}} \leq \sum_{n=1}^{M} \frac{1}{2^n} \tag{1}$$

for all $M \in \mathbb{N}$, because in general the numbers $\{\phi_A(n)\colon n = 1, \ldots, M\}$ differ from the numbers $\{1, \ldots, M\}$, which maximize the sum on the right-hand side of (1). Therefore, we see that

$$\sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}} \leq \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \tag{2}$$

Hence $\{s_M\}_{M\in M}$ is a monotonically increasing and bounded sequence of real numbers. According to the monotone convergence theorem, this sequence has a well-defined limit

$$s^* = \lim_{M\to\infty} s_M = \sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}},$$

where $s^* \in \mathbb{R}$. However it can be shown that $s^* \notin \mathbb{R}_c$, i.e., s^* is not computable [25, Corollary 2b, p. 20]. This fact will be important for us.

A sequence $\{x(k)\}_{k\in\mathbb{Z}}$ in ℓ^p, $p \in [1, \infty) \cap \mathbb{R}_c$ is called computable in ℓ^p if every number $x(k)$, $k \in \mathbb{Z}$, is computable and there exist a computable sequence $\{y_n\}_{n\in\mathbb{N}} \subset \ell^p$, where each y_n has only finitely many non-zero elements and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have $\|x - y_n\|_{\ell^p} \leq 2^{-M}$ for all $n \geq \xi(M)$. By $\mathcal{C}\ell^p$ we denote the set of all sequences that are computable in ℓ^p. Similarly, we define the set of all sequences that are computable in c_0 and denote this set by $\mathcal{C}c_0$.

There are several—not equivalent—definitions of computable functions, most notably, computable continuous functions, Turing computable functions, Markov computable functions, and Banach–Mazur computable functions [26]. A function that is computable with respect to any of the above definitions has the property that it maps computable numbers into computable numbers.

We now give a definition of a computable continuous function. Let $I \subset \mathbb{R}$ be an interval where the endpoints are computable real numbers. A function $f\colon I \to \mathbb{R}$ is called a computable continuous function if

1) f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset I$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of real numbers,
2) there exists a recursive function $d\colon \mathbb{N} \to \mathbb{N}$, such that for all $t_1, t_2 \in I$ and all $M \in \mathbb{N}$ we have: $|t_1 - t_2| \leq 1/d(M)$ implies $|f(t_1) - f(t_2)| \leq 2^{-M}$.

We extend this definition to functions defined on $\mathbb{R}$. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called computable continuous function if

1) f maps every computable sequence $\{t_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ into a computable sequence $\{f(t_n)\}_{n\in\mathbb{N}}$ of real numbers.
2) there exists a recursive function $d\colon \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ such that for all $L, M \in \mathbb{N}$ we have: $|t_1 - t_2| \leq 1/d(L, M)$ implies $|f(t_1) - f(t_2)| \leq 2^{-M}$ for all $t_1, t_2 \in [-L, L]$.

In addition to the definition of computability for continuous functions as given above, we introduce a definition for computable functions in Banach spaces, which is based on effective convergence. We call a function f elementary computable if there exists a natural number L and a sequence of computable numbers $\{\alpha_k\}_{k=-L}^{L}$ such that

$$f(t) = \sum_{k=-L}^{L} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{3}$$

Note that every elementary computable function f is a finite sum of computable continuous functions and hence a computable continuous function. As a consequence, for every $t \in \mathbb{R}_c$ the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is an elementary computable function, as well as the product of an elementary computable function with a computable number.

A function in $f \in \mathcal{B}_\pi^p$ is called computable in $\mathcal{B}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, if there exist a computable sequence of elementary computable functions $\{f_N\}_{N\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $M \in \mathbb{N}$ we have $\|f - f_N\|_{\mathcal{B}_\pi^p} \leq 2^{-M}$ for all $N \geq \xi(M)$. By $\mathcal{CB}_\pi^p$, $p \in [1, \infty) \cap \mathbb{R}_c$, we denote the set of all functions in $\mathcal{B}_\pi^p$ that are computable in $\mathcal{B}_\pi^p$. The set $\mathcal{CB}_{\pi,0}^\infty$ of computable functions in $\mathcal{B}_{\pi,0}^\infty$ is defined analogously. Note that the sets $\mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $\mathcal{CB}_{\pi,0}^\infty$ are non-empty. Consider, for example, the function

$$f(t) = \frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi(t+2k))}{\pi(t+2k)} = \frac{2k\sin(\pi t)}{\pi t(t+2k)}.$$

Clearly f is an elementary computable function. Further, since $f \in \mathcal{B}_\pi^p$ for all $p \in [1,\infty)$ and $f \in \mathcal{B}_{\pi,0}^\infty$, it follows that $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and $f \in \mathcal{CB}_{\pi,0}^\infty$.

In order that the above definition of a computable function in $\mathcal{B}_\pi^1$ makes sense, it is necessary that each $f \in \mathcal{B}_\pi^1$ can be approximated in a classical sense by a linear combination of shifted sinc-functions. This is assured by the next fact, the proof of which will be given in Appendix B.

Fact 1. *Let $f \in \mathcal{B}_\pi^1$. For every $\epsilon > 0$ there exist an $N \in \mathbb{N}$ and numbers $\{\alpha_k\}_{k=-N}^N$ such that*[1]

$$\left\| f - \sum_{k=-N}^{N} \alpha_k \frac{\sin(\pi(\,\cdot\, - k))}{\pi(\,\cdot\, - k)} \right\|_{\mathcal{B}_\pi^1} < \epsilon.$$

Remark 1. If $f \in \mathcal{CB}_\pi^p$ for $p \in [1,\infty) \cap \mathbb{R}_c$ or $f \in \mathcal{CB}_{\pi,0}^\infty$ for $p = \infty$, then the norm $\|f\|_{\mathcal{B}_\pi^p}$ is computable. This follows from the fact that the norm $\|f_N\|_{\mathcal{B}_\pi^p}$ of an elementary computable function f_N is computable, together with the inequality

$$\left| \|f\|_{\mathcal{B}_\pi^p} - \|f_N\|_{\mathcal{B}_\pi^p} \right| \le \|f - f_N\|_{\mathcal{B}_\pi^p} \le \frac{1}{2^M},$$

which holds for all $N \ge \xi(M)$.

Remark 2. For $p \in [1,\infty)$ and signals $f \in \mathcal{B}_\pi^p$, it follows from Nikol'skiĭ's inequality [32, p. 49] that $\|f\|_\infty \le (1+\pi)\|f\|_{\mathcal{B}_\pi^p}$. Hence for $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, and all $M \in \mathbb{N}$ we have

$$\|f - f_N\|_\infty \le (1+\pi)\|f - f_N\|_{\mathcal{B}_\pi^p} \le \frac{1+\pi}{2^M}$$

for all $N \ge \xi(M)$. This shows that we can approximate any function $f \in \mathcal{CB}_\pi^p$ by an elementary computable function where we have an "effective" and uniform control of the approximation error, as illustrated in Fig. 1.

Remark 3. If $f \in \mathcal{CB}_\pi^p$, $p \in [1,\infty) \cap \mathbb{R}_c$, then f is also a computable continuous function, because we have

$$\begin{aligned} |f(t_1) - f(t_2)| &\le \|f'\|_\infty |t_1 - t_2| \\ &\le \pi \|f\|_\infty |t_1 - t_2| \\ &\le \pi(1+\pi)\|f\|_{\mathcal{B}_\pi^p} |t_1 - t_2|, \end{aligned}$$

and $\|f\|_{\mathcal{B}_\pi^p}$ is computable.

Remark 4. For our definition of a computable function in $\mathcal{B}_\pi^p$, we employ finite Shannon sampling series, as given by (3), as basic building blocks. This corresponds to equidistant sampling at the Nyquist rate. It is possible to extend the questions in this paper to non-equidistant sampling.

In the theory of non-equidistant sampling, the generating function ϕ plays a central role. If the sampling point sequences $\{t_k\}_{k\in\mathbb{Z}} \subset \mathbb{R}$ is ordered strictly increasingly, then the product

$$\phi(z) = z \lim_{N\to\infty} \prod_{\substack{|k|\le N \\ k\ne 0}} \left(1 - \frac{z}{t_k}\right) \tag{4}$$

[1]The "$\cdot$" in the function argument is a placeholder for the anonymous variable with respect to which the norm is taken.

converges uniformly on $|z| \le R$ for all $R < \infty$, and ϕ is an entire function of exponential type π [42]. Without loss of generality, we assumed that $t_0 = 0$. It can be seen from (4) that ϕ has the zeros $\{t_k\}_{k\in\mathbb{Z}}$. Based on ϕ, we can define the interpolation functions

$$\phi_k(t) = \frac{\phi(t)}{\phi'(t_k)(t - t_k)}, \quad k \in \mathbb{Z},$$

which are the unique functions in $\mathcal{B}_\pi^2$ that solve the interpolation problem $\phi_k(t_l) = 1$ if $k = l$ and $\phi_k(t_l) = 0$ if $k \ne l$. Under certain conditions on the sequence of sampling points $\{t_k\}_{k\in\mathbb{Z}}$ and the signal f, we can use the sampling series

$$\sum_{k=-\infty}^{\infty} f(t_k)\phi_k(t)$$

to reconstruct f. However, when studying non-equidistant sampling, new problems emerge. Even if the sampling points $\{t_k\}_{k\in\mathbb{Z}}$ are computable numbers, it is unclear whether the generating function ϕ is a computable entire function. To the best of our knowledge, it seems to be completely unknown how to infer the computability of ϕ from properties of the computable sequence $\{t_k\}_{k\in\mathbb{Z}}$.

Furthermore, it would be interesting to analyze the effective approximation of bandlimited signals by sampling series for the case where oversampling is employed. As before, no results seem to exist for this question.

III. A Necessary and Sufficient Condition

We start with a simple but important observation about the computability of the discrete-time signal: The computability of the continuous-time bandlimited signal always implies the computability of the corresponding discrete-time signal that is obtained by sampling at the Nyquist rate.

Observation 1. *Let $p \in [1,\infty) \cap \mathbb{R}_c$ or $p = \infty$, and let $f \in \mathcal{CB}_\pi^p$ if $p \in [1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{CB}_{\pi,0}^\infty$ if $p = \infty$. Then $f|_\mathbb{Z} = \{f(k)\}_{k\in\mathbb{N}}$ is a computable sequence of computable numbers. Further, we have $f|_\mathbb{Z} \in \mathcal{C}\ell^p$ if $p \in [1,\infty) \cap \mathbb{R}_c$, and $f|_\mathbb{Z} \in \mathcal{C}c_0$ if $p = \infty$.*

For the proof of Observation 1 we need the Plancherel–Pólya inequality as a lemma. This inequality connects the L^p-norm of a continuous-time bandlimited signal with the ℓ^p-norm of the sequence of its samples [42, p. 152]. In Appendix A we will show how it can be used to prove the convergence of the Shannon sampling series.

Lemma 1 (Plancherel–Pólya)**.** *Let $1 < p < \infty$. There exist two positive constants A_p and B_p such that for all $f \in \mathcal{B}_\pi^p$ we have*

$$A_p \left[\sum_{k=-\infty}^{\infty} |f(k)|^p \right]^{\frac{1}{p}} \le \left[\int_{-\infty}^{\infty} |f(t)|^p \,\mathrm{d}t \right]^{\frac{1}{p}} \le B_p \left[\sum_{k=-\infty}^{\infty} |f(k)|^p \right]^{\frac{1}{p}}.$$

This inequality and the fact that the constants A_p and B_p are computable if $p \in \mathbb{R}_c$ will be essential for our proofs. The value of B_p was derived in [34], and for A_p we can choose $A_p = 1/(1+\pi)$, according to Nikol'skiĭ's inequality [32, p. 49].

Proof of Observation 1. Let $p \in [1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{CB}_\pi^p$ be arbitrary but fixed. Since f is computable in $\mathcal{B}_\pi^p$, there exists a computable sequence of elementary computable functions $\{f_N\}_{N\in\mathbb{N}}$ that converges effectively to f. That is for all $M \in \mathbb{N}$, we have

$$\|f - f_N\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^M}$$

for all $N \geq \xi(M)$. It follows from Lemma 1 that

$$\|f|_{\mathbb{Z}} - f_N|_{\mathbb{Z}}\|_{\ell^p} \leq \frac{1}{A_p}\|f - f_N\|_{\mathcal{B}_\pi^p} < \frac{1}{A_p 2^M}$$

for all $N \geq \xi(M)$, which shows that $f|_{\mathbb{Z}} \in \mathcal{C}\ell^p$. The case $p = \infty$ and $f \in \mathcal{B}_{\pi,0}^\infty$ is treated analogously, using that $\|f|_{\mathbb{Z}}\|_{\ell^\infty} \leq \|f\|_{\mathcal{B}_{\pi,0}^\infty}$ for all $f \in \mathcal{B}_{\pi,0}^\infty$. □

In Observation 1 we have seen that the computability of the continuous-time signal f directly carries over to the computability of the discrete-time signal $f|_{\mathbb{Z}}$, which is obtained by sampling f. Clearly, the other direction is relevant as well. If we have a computable discrete-time signal, is the corresponding continuous-time signal computable? Such a question also arises in many modern applications, where we do not start with an analog signal but instead with a digital signal, such as a synthetically created digital image, audio sample, or baseband signal, that is later converted into an analog continuous-time signal. Then we need to approximate the continuous-time signal by using the discrete-time signal while controlling the approximation error. Such a control of the approximation error is possible only if the desired continuous-time signal is computable.

A second problem is related to the actual computation of the continuous-time signal. Even if we know that the continuous-time signal is computable, we do not necessarily have a simple algorithm to compute it. Having such an algorithm is clearly essential for applications.

Hence, two questions are important. Question 1: Is there a simple necessary and sufficient condition for characterizing the computability of f? And, Question 2: Is there a simple canonical algorithm to actually compute f from the samples $f|_{\mathbb{Z}}$? We will analyze both questions and prove that for $p \in (1,\infty) \cap \mathbb{R}_c$ and signals $f \in \mathcal{B}_\pi^p$, they can be answered with "yes". For $p = 1$ and $p = \infty$ this is not possible.

We start with Question 1. Theorem 1 shows that for $p \in (1,\infty) \cap \mathbb{R}_c$, the computability of the discrete-time signal implies the computability of the continuous-time signal. This answers Question 1 for $p \in (1,\infty) \cap \mathbb{R}_c$, because $f|_{\mathbb{Z}}$, i.e., the samples of f provide a simple discrete-time representation of the signal f, where computability in one domain implies computability in the other domain.

Theorem 1. *Let $p \in (1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if $f|_{\mathbb{Z}} \in \mathcal{C}\ell^p$.*

Proof. "$\Rightarrow$": This follows directly from Observation 1.

"$\Leftarrow$": Let $p \in (1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$ be arbitrary but fixed. Further, let $f|_{\mathbb{Z}} \in \mathcal{C}\ell^p$. Then there exist a computable sequence $\{\alpha(N)\}_{N\in\mathbb{N}}$ of sequences with only finitely many non-zero elements

$$\alpha(N) = \{\ldots, 0, \alpha_{-K(N)}(N), \ldots, \alpha_{K(N)}(N), 0, \ldots\}$$

and a recursive function ξ, such that for all $M \in \mathbb{N}$, we have

$$\|f|_{\mathbb{Z}} - \alpha(N)\|_{\ell^p} \leq \frac{1}{2^M}$$

for all $N \geq \xi(M)$. Let

$$f_N(t) = \sum_{k=-L(N)}^{L(N)} \alpha_k(N) \frac{\sin(\pi(t-k))}{\pi(t-k)}.$$

Then $\{f_N\}_{n\in\mathbb{N}}$ is a computable sequence of elementary computable functions in $\mathcal{B}_\pi^p$, and using Lemma 1 we obtain, for all $M \in \mathbb{N}$, that

$$\|f - f_N\|_{\mathcal{B}_\pi^p} \leq B_p \|f|_{\mathbb{Z}} - \alpha(N)\|_{\ell^p} \leq B_p \frac{1}{2^M}$$

for all $N \geq \xi(M)$. Hence we can compute an integer $\overline{M}$ which can depend on p, such that $B_p \leq 2^{\overline{M}}$. Let $\overline{\xi}(M) = \xi(M + \overline{M})$. Then for all $M \in \mathbb{N}$, we have

$$\|f - f_N\|_{\mathcal{B}_\pi^p} \leq \frac{1}{2^M}$$

for all $N \geq \overline{\xi}(M)$. This shows that $f \in \mathcal{CB}_\pi^p$. □

According to Theorem 1, we have a correspondence between the computable discrete-time signals in $\mathcal{C}\ell^p$ and the computable continuous-time signals in $\mathcal{CB}_\pi^p$ for $p \in (1,\infty) \cap \mathbb{R}_c$. This correspondence is no longer true for $p = 1$ as the next observation shows.

Observation 2. *Let $f(t) = \sin(\pi t)/(\pi t)$, $t \in \mathbb{R}$. Then f is an entire function of exponential type at most π, and we have $f|_{\mathbb{Z}} \in \mathcal{C}\ell^1$. However, $f \notin \mathcal{CB}_\pi^1$, because $f \notin \mathcal{B}_\pi^1$.*

IV. The Shannon Sampling Series as a Canonical Algorithm

In this section we will answer Question 2 by showing that the Shannon sampling series provides a canonical algorithm for the effective approximation of $f \in \mathcal{CB}_\pi^p$ if $p \in (1,\infty) \cap \mathbb{R}_c$. This gives us a remarkably simple algorithm to construct a computable sequence of elementary computable functions in $\mathcal{CB}_\pi^p$ that converges effectively to f. For $N \in \mathbb{N}$, let

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

Theorem 2. *Let $p \in (1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$. Then we have $f \in \mathcal{CB}_\pi^p$ if and only if $f|_{\mathbb{Z}}$ is a computable sequence of computable numbers and $\{S_N f\}_{N\in\mathbb{N}}$ converges effectively to f in the L^p-norm.*

For the proof of Theorem 2 we need the following lemma [25, p. 20, Corollary 2a].

Lemma 2. *Let $\{x_N\}_{N\in\mathbb{N}}$ be a computable sequence of computable numbers satisfying $x_N \leq x_{N+1}$, $N \in \mathbb{N}$, and $x_* = \lim_{N\to\infty} x_N$ with $x_* \in \mathbb{R}_c$. Then there exists a recursive function ξ such that for all $M \in \mathbb{N}$ we have $|x_* - x_N| \leq 1/2^M$ for all $N \geq \xi(M)$.*

Proof of Theorem 2. "$\Leftarrow$": This direction is obvious. $\{S_N f\}_{N\in\mathbb{N}}$ is a computable sequence of elementary computable functions that converges effectively to f in the

L^p-norm. The computability of f follows immediately from the definition of $\mathcal{CB}_\pi^p$.

"$\Rightarrow$": Let $p \in (1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$ be arbitrary but fixed. For $N \in \mathbb{N}$, let $\alpha(N) = \{\alpha_k(N)\}_{k\in\mathbb{Z}}$, where

$$\alpha_k(N) = \begin{cases} f(k), & |k| \le N \\ 0, & |k| > N. \end{cases}$$

Then we have

$$\lim_{N\to\infty} \|f|_{\mathbb{Z}} - \alpha(N)\|_{\ell^p} = 0$$

as well as

$$\lim_{N\to\infty} \|\alpha(N)\|_{\ell^p} = \|f|_{\mathbb{Z}}\|_{\ell^p}.$$

Since $f \in \mathcal{CB}_\pi^p$, we already know from Observation 1 that $f|_{\mathbb{Z}} \in \mathcal{C}\ell^p$. Further, since

$$\|\alpha(N)\|_{\ell^p}^p = \sum_{k=-N}^{N} |f(k)|^p$$

is monotonically increasing in N, it follows from Lemma 2 that there exists a recursive function ξ, such that for all $M \in \mathbb{N}$, we have

$$|\|f|_{\mathbb{Z}}\|_{\ell^p}^p - \|\alpha(N)\|_{\ell^p}^p| \le \frac{1}{2^M}$$

for all $N \ge \xi(M)$. Consequently, we have

$$\begin{aligned} \|f|_{\mathbb{Z}} - \alpha(N)\|_{\ell^p}^p &= \left| \sum_{|k|>N} |f(k)|^p \right| \\ &= \left| \sum_{k=-\infty}^{\infty} |f(k)|^p - \sum_{k=-N}^{N} |\alpha_k(N)|^p \right| \\ &= |\|f|_{\mathbb{Z}}\|_{\ell^p}^p - \|\alpha(N)\|_{\ell^p}^p| \le \frac{1}{2^M} \end{aligned}$$

for all $N \ge \xi(M)$. Hence the sequence $\{\alpha(N)\}_{N\in\mathbb{N}}$ converges effectively to $f|_{\mathbb{Z}}$ in the ℓ^p-norm. We set $\overline{\xi}(M) = \xi((M + \overline{M})\overline{p})$, where $\overline{M}$ is such that $B_p < 2^{\overline{M}}$, and $\overline{p}$ is the smallest integer satisfying $\overline{p} > p$. Then we have

$$\|f - S_N f\|_{\mathcal{B}_\pi^p} \le B_p \|f|_{\mathbb{Z}} - \alpha(N)\|_{\ell^p} < \frac{1}{2^M}$$

for all $N \ge \overline{\xi}(M)$. □

The next two theorems show that Theorem 2 cannot be true for $p = 1$ and $p = \infty$, respectively. In particular, they show that in general, the Shannon sampling series does not provide an effective approximation process for $\mathcal{B}_\pi^1$ and $\mathcal{B}_{\pi,0}^\infty$.

Theorem 3. *There exists a signal $f_1 \in \mathcal{CB}_\pi^1$ for which $S_1 f_1 \notin \mathcal{CB}_\pi^1$ because $S_1 f_1 \notin \mathcal{B}_\pi^1$.*

Proof. We choose

$$\begin{aligned} f_1(t) &= \frac{\sin(\pi t)}{\pi t} + \frac{1}{2}\left(\frac{\sin(\pi(t+3))}{\pi(t+3)} + \frac{\sin(\pi(t-3))}{\pi(t-3)} \right) \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} (1 + \cos(3\omega))\, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega. \end{aligned}$$

Then the Fourier transform of f_1 is given by

$$\hat{f}_1(\omega) = \begin{cases} 1 + \cos(3\omega), & |\omega| \le \pi, \\ 0, & |\omega| > \pi, \end{cases}$$

and its derivative by

$$\hat{f}_1'(\omega) = \begin{cases} -3\sin(3\omega), & |\omega| \le \pi, \\ 0, & |\omega| > \pi. \end{cases}$$

Since $\hat{f}_1'$ is bounded on $[-\pi,\pi]$ and zero otherwise, it follows that $\hat{f}_1' \in L^2(\mathbb{R})$, which in turn implies that $f_1 \in \mathcal{CB}_\pi^1$. Further, we have $(S_1 f_1)(t) = \sin(\pi t)/(\pi t)$, and therefore $S_1 f_1 \notin \mathcal{B}_\pi^1$. □

Theorem 4. *There exists a signal $f_2 \in \mathcal{CB}_{\pi,0}^\infty$ such that $\{S_N f_2\}_{N\in\mathbb{N}}$ does not converge effectively to f_2 in the L^∞-norm.*

Proof. Let

$$w_N(k) = \begin{cases} 1, & |k| \le N, \\ 1 - \frac{|k|-N}{N}, & N < |k| \le 2N, \\ 0, & |k| > 2N, \end{cases}$$

and

$$\begin{aligned} g_N(t) &= \sum_{k=-2N}^{2N} (-1)^k w_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} \left(\sum_{k=-2N}^{2N} (-1)^k w_N(k)\, \mathrm{e}^{-i\omega k} \right) \mathrm{e}^{i\omega t} \, \mathrm{d}\omega. \end{aligned}$$

Further, let

$$W_N(\omega) = \sum_{k=-2N}^{2N} w_N(k)\, \mathrm{e}^{-i\omega k}.$$

It can be shown that $1/(2\pi) \int_{-\pi}^{\pi} |W_N(\omega)| \, \mathrm{d}\omega < 3$, see for example [43]. Since

$$\begin{aligned} W_N(\omega + \pi) &= \sum_{k=-2N}^{2N} w_N(k)\, \mathrm{e}^{-i\omega k}\, \mathrm{e}^{-i\pi k} \\ &= \sum_{k=-2N}^{2N} (-1)^k w_N(k)\, \mathrm{e}^{-i\omega k}, \end{aligned}$$

it follows that

$$\begin{aligned} |g_N(t)| &\le \frac{1}{2\pi} \int_{-\pi}^{\pi} |W_N(\omega + \pi)| \, \mathrm{d}\omega \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |W_N(\omega)| \, \mathrm{d}\omega < 3, \end{aligned} \tag{5}$$

where we used the fact that W_N is 2π-periodic. We further have

$$\begin{aligned} (S_N g_N)(t) &= \sum_{k=-N}^{N} (-1)^k w_N(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \\ &= \frac{\sin(\pi t)}{\pi} \sum_{k=-N}^{N} \frac{1}{t-k}. \end{aligned}$$

For $t = N + 1/2$ it follows that

$$
\begin{aligned}
|(S_N g_N)(N+\tfrac{1}{2})| &= \frac{1}{\pi}\sum_{k=-N}^{N}\frac{1}{N+\frac{1}{2}-k} \\
&= \frac{1}{\pi}\sum_{k=0}^{2N}\frac{1}{k+\frac{1}{2}} > \frac{1}{\pi}\sum_{k=0}^{2N}\int_k^{k+1}\frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau \\
&= \frac{1}{\pi}\int_0^{2N+1}\frac{1}{\tau+\frac{1}{2}}\,\mathrm{d}\tau = \frac{1}{\pi}\log(4N+3). \qquad (6)
\end{aligned}
$$

Let $C_N = |(S_N g_N)(N+\frac{1}{2})|$. Note that C_N is monotonically increasing in N, and that, for each $N \in \mathbb{N}$, C_N is a computable number.

Next we construct the function f_2. Let $A \subset \mathbb{N}$ be a recursively enumerable non-recursive set and $\phi_A \colon \mathbb{N} \to A$ a recursive enumeration of A. We set

$$q_1(t) = \frac{g_1(t)}{2^{\phi_A(1)} C_1}.$$

Let $\overline{N}_1 = 2$, $k_1 = 0$, $N_1^{(1)} = -1$, and $N_1^{(2)} = 1$. We set $\overline{N}_2 = \overline{N}_1 + 4\cdot 2$, $k_2 = \overline{N}_1 + 2\cdot 2$, $N_2^{(1)} = k_2 - 2$, $N_2^{(2)} = k_2 + 2$, and

$$q_2(t) = q_1(t) + \frac{g_2(t-k_2)}{2^{\phi_A(2)} C_2}.$$

Assume that for some $r \in \mathbb{N}$, we have already constructed $\overline{N}_r$, k_r, $N_r^{(1)}$, $N_r^{(2)}$, and q_r. Then we set $\overline{N}_{r+1} = \overline{N}_r + 4(r+1)$, $k_{r+1} = \overline{N}_r + 2(r+1)$, $N_{r+1}^{(1)} = k_{r+1} - (r+1)$, $N_{r+1}^{(2)} = k_{r+1} + (r+1)$ and

$$q_{r+1}(t) = q_r(t) + \frac{g_{r+1}(t-k_{r+1})}{2^{\phi_A(r+1)} C_{r+1}}.$$

For $r \in \mathbb{N}$ we have $q_r \in \mathcal{B}_{\pi,0}^{\infty}$, as well as $q_r \in \mathcal{CB}_{\pi,0}^{\infty}$, because q_r is an elementary computable function. It can be shown that $\{q_r\}_{r\in\mathbb{N}}$ forms a Cauchy sequence in $\mathcal{B}_{\pi,0}^{\infty}$. Thus, the limit

$$f_2(t) = \lim_{r\to\infty} q_r(t) = \sum_{r=1}^{\infty}\frac{g_r(t-k_r)}{2^{\phi_A(r)} C_r}, \quad t \in \mathbb{R},$$

exists, and f_2 is a function in $\mathcal{B}_{\pi,0}^{\infty}$. Further, we have

$$
\begin{aligned}
&\left\| f_2 - \sum_{r=1}^{N}\frac{g_r(\,\cdot\,-k_r)}{2^{\phi_A(r)} C_r}\right\|_{\mathcal{B}_{\pi,0}^{\infty}} = \left\| \sum_{r=N+1}^{\infty}\frac{g_r(\,\cdot\,-k_r)}{2^{\phi_A(r)} C_r}\right\|_{\mathcal{B}_{\pi,0}^{\infty}} \\
&\le \sum_{r=N+1}^{\infty}\frac{\|g_r(\,\cdot\,-k_r)\|_{\mathcal{B}_{\pi,0}^{\infty}}}{2^{\phi_A(r)} C_r} < \frac{3}{C_{N+1}}\sum_{r=N+1}^{\infty}\frac{1}{2^{\phi_A(r)}} \\
&< \frac{3}{C_{N+1}} < \frac{3\pi}{\log(4N+3)},
\end{aligned}
$$

where we used (5) and the monotonicity of C_N in the second inequality, (2) in the third inequality, and (6) in the last inequality. This shows that f_2 is computable in $\mathcal{B}_{\pi,0}^{\infty}$, i.e., that $f_2 \in \mathcal{CB}_{\pi,0}^{\infty}$.

The rest of the proof is done indirectly. We assume that there exists a recursive function ξ_2, such that for all $M \in \mathbb{N}$, we have

$$\|f_2 - S_N f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} \le \frac{1}{2^M}$$

for all $N \ge \xi_2(M)$, and prove that this assumption leads to a contradiction. Let $M \in \mathbb{N}$ be arbitrary but fixed. According to the assumption we have

$$
\begin{aligned}
\|S_{N_1} f_2 - S_{N_2} f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} &= \|S_{N_1} f_2 - f_2 + f_2 - S_{N_2} f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} \\
&\le \|f_2 - S_{N_1} f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} + \|f_2 - S_{N_2} f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} \le \frac{2}{2^M}
\end{aligned}
$$

for all $N_1, N_2 \ge \xi_2(M)$. Let r_0 be the smallest natural number such that $\overline{N}_{r_0} > \xi_2(M)$, and let $r > r_0$. We have

$$
\begin{aligned}
&\|S_{N_r^{(2)}} f_2 - S_{N_r^{(1)}-1} f_2\|_{\mathcal{B}_{\pi,0}^{\infty}} \\
&= \max_{t\in\mathbb{R}}\left|\sum_{k=N_r^{(1)}}^{N_r^{(2)}} f_2(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\
&= \frac{1}{2^{\phi_A(r)} C_r}\max_{t\in\mathbb{R}}\left|\sum_{k=N_r^{(1)}}^{N_r^{(2)}} g_r(k-k_r)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\
&= \frac{1}{2^{\phi_A(r)} C_r}\max_{t\in\mathbb{R}}\left|\sum_{k=-r}^{r} g_r(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right| \\
&\ge \frac{1}{2^{\phi_A(r)} C_r}\left|\sum_{k=-r}^{r} g_r(k)\frac{\sin(\pi(r+\frac{1}{2}-k))}{\pi(r+\frac{1}{2}-k)}\right| \\
&= \frac{1}{2^{\phi_A(r)}}.
\end{aligned}
$$

Since $N_r^{(1)} > \overline{N}_{r_0}$ and $N_r^{(2)} > \overline{N}_{r_0}$, it follows that $1/2^{\phi_A(r)} < 2/2^M$ for all $r > r_0$, and consequently that $M - 1 < \phi_A(r)$. The last inequality is valid for all $M \in \mathbb{N}$ and all $r \in \mathbb{N}$, satisfying $r > r_0$ and $\overline{N}_{r_0} > \xi_2(M)$.

Let $s \in \mathbb{N}$ be arbitrary but fixed. We will give an algorithm that can decide for every natural number in the interval $[1, s]$ if it belongs to A or to $A^{\complement}$. To this end, we determine $\xi_2(s+1)$. Let r_0 be the smallest natural number such that $\overline{N}_{r_0} > \xi_2(s+1)$. Then for all $r \in \mathbb{N}$ with $r > r_0$ we have $s = (s+1) - 1 < \phi_A(r)$. Hence the sequence $\{\phi_A(r)\}_{r=r_0+1}^{\infty}$ will not hit the interval $[1, s]$. Let $A_s = \{\phi_A(r)\}_{r=1}^{r_0}$. Then we have $A_s \subset A$. Let $\underline{A}_s = A_s \cap [1, s]$. We have $[1, s] \setminus \underline{A}_s \subset A^{\complement}$, because for $k \in [1, s] \setminus \underline{A}_s$, we have $k \notin A_s$, and for $r > \xi_2(s+1)$, k is not hit by $\phi_A(r)$, i.e., we have $k \in A^{\complement}$. Since $s \in \mathbb{N}$ was arbitrary, we have an algorithm that, for all $k \in \mathbb{N}$, can decide whether $k \in A$ or $k \in A^{\complement}$. This shows that A is recursive, which is a contradiction. Thus, our assumption that $\{S_N f_2\}_{N\in\mathbb{N}}$ converges effectively to f_2 in the L^{∞}-norm has to be wrong. □

V. Behavior in Discrete and Continuous Time

A. Case $1 < p < \infty$

We already know that bandlimited signals in $\mathcal{B}_{\pi}^{p}$ are not necessarily in $\mathcal{CB}_{\pi}^{p}$. For a given signal $f \in \mathcal{B}_{\pi}^{p}$ to be computable, it is necessary to find an algorithm that effectively approximates f using a computable sequence of elementary computable functions, i.e. a computable sequence of finite sampling series. It would be desirable to have a test that can decide whether such an approximation exists, and ideally, if the answer is positive, can derive this effective approximation algorithm from the signal f itself.

Especially useful would be a test that can decide the membership of f to $\mathcal{CB}_\pi^p$ based on the samples of f. As we will show next in Theorem 5, for $p \in (1,\infty) \cap \mathbb{R}_c$, such a test is possible. We have a simple necessary and sufficient condition for the computability of the continuous-time signal that is based solely on properties of the sequence of samples $f|_\mathbb{Z}$. Compared to Theorem 1, the condition is even simpler, as we do not require the computability of $f|_\mathbb{Z}$ in ℓ^p, but only the computability of the number $\|f|_\mathbb{Z}\|_{\ell^p}$.

Theorem 5. *Let $p \in (1,\infty) \cap \mathbb{R}_c$ and $f \in \mathcal{B}_\pi^p$. We have $f \in \mathcal{CB}_\pi^p$ if and only if*

1) $f|_\mathbb{Z}$ is a computable sequence of computable numbers,
2) $\|f|_\mathbb{Z}\|_{\ell^p} \in \mathbb{R}_c$.

Remark 5. Note that in the theorem we only require that $\|f|_\mathbb{Z}\|_{\ell^p}$ be a computable number. We do not require that the number $\|f|_\mathbb{Z}\|_{\ell^p}$ be computed from f or $f|_\mathbb{Z}$.

Proof of Theorem 5. "$\Rightarrow$": Item 1) follows directly from Observation 1. Further, item 2) of the theorem follows from Lemma 1.

"$\Leftarrow$": This direction is proved using the same arguments that were used in the proof of Theorem 2. If $\|f|_\mathbb{Z}\|_{\ell^p} \in \mathbb{R}_c$, then the sequence that has been constructed in the proof of Theorem 2 converges effectively to $f|_\mathbb{Z}$ in the ℓ^p-norm. This implies that the Shannon sampling series $S_N f$ converges effectively to f in the L^p-norm, which in turn implies the computability of f in $\mathcal{B}_\pi^p$, according to Theorem 2. □

The necessary and sufficient condition for computability which was given in Theorem 5 is true for all $p \in (1,\infty) \cap \mathbb{R}_c$. However, it cannot be extended to hold for $p = 1$, as we will see in the next section.

B. Case $p = 1$

In the last section we have seen that for $p \in (1,\infty) \cap \mathbb{R}_c$, the effective approximation of a discrete-time signal in ℓ^p is directly coupled to the effective approximation of the corresponding continuous-time signal in $\mathcal{B}_\pi^p$. In this section we will show that this result cannot be extended to hold for $p = 1$.

We will construct a discrete-time signal that we can effectively approximate in the ℓ^1-norm, and for which the corresponding bandlimited continuous-time signal is in $\mathcal{B}_\pi^1$ but the $\mathcal{B}_\pi^1$-norm is not computable. This implies, according to Remark 1, that the bandlimited continuous-time signal cannot be effectively approximated by elementary bandlimited functions in $\mathcal{B}_\pi^1$.

We will also use this result in Section VII-A to study the BIBO stability norm of discrete-time and continuous-time LTI systems.

Theorem 6. *There exists a function $f_3 \in \mathcal{B}_\pi^1$ such that*

1) $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$,
2) $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$.

We postpone the proof of Theorem 6.

Remark 6. Note that $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$ implies that $f_3|_\mathbb{Z}$ is a computable sequence of computable numbers and that the ℓ^1-norm of $f_3|_\mathbb{Z}$ is computable.

If we have a signal $f_3 \in \mathcal{B}_\pi^1$ that satisfies $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$, such as the signal f_3 in Theorem 6, then we also have $f_3|_\mathbb{Z} \in \mathcal{C}\ell^p$ for all $p \in (1,\infty)$, as we will show next. First, we note that $f_3|_\mathbb{Z} \in \ell^1$ implies that $f_3|_\mathbb{Z} \in \ell^p$ for all $p \in (1,\infty)$. Further, we have

$$\begin{aligned}\frac{(\|f_3|_\mathbb{Z}\|_{\ell^p})^p}{(\|f_3|_\mathbb{Z}\|_{\ell^1})^p} - \frac{\sum_{|k|\le N}|f_3(k)|^p}{(\|f_3|_\mathbb{Z}\|_{\ell^1})^p} &\le \sum_{|k|>N}\left|\frac{f_3(k)}{\|f_3|_\mathbb{Z}\|_{\ell^1}}\right|^p \\ &\le \sum_{|k|>N}\frac{|f_3(k)|}{\|f_3|_\mathbb{Z}\|_{\ell^1}},\end{aligned}$$

where we used in the second inequality that

$$\left|\frac{f_3(k)}{\|f_3|_\mathbb{Z}\|_{\ell^1}}\right|^p \le \left|\frac{f_3(k)}{\|f_3|_\mathbb{Z}\|_{\ell^1}}\right|$$

for all $k \in \mathbb{Z}$, which holds because

$$\frac{|f_3(k)|}{\|f_3|_\mathbb{Z}\|_{\ell^1}} \le 1.$$

Since $\|f_3|_\mathbb{Z}\|_{\ell^1}$ is a computable number, and $\{\sum_{|k|\le N}|f_3(k)|\}_{N\in\mathbb{N}}$ is monotonically increasing and converges to $\|f_3|_\mathbb{Z}\|_{\ell^1}$, it follows from Lemma 2 that $\{\sum_{|k|>N}|f_3(k)|\}_{N\in\mathbb{N}}$ converges effectively to zero. This shows that for $p \in (1,\infty) \cap \mathbb{R}_c$, the computable sequence of computable numbers $\{(\sum_{|k|\le N}|f_3(k)|^p)^{1/p}\}_{N\in\mathbb{N}}$ converges effectively to $\|f_3|_\mathbb{Z}\|_{\ell^p}$. Hence we have $\|f_3|_\mathbb{Z}\|_{\ell^p} \in \mathbb{R}_c$ and consequently, $f_3 \in \mathcal{CB}_\pi^p$ for all $p \in (1,\infty) \cap \mathbb{R}_c$ because of Theorem 5. This also implies that

$$\max_{t\in\mathbb{R}}\left|f_3(t) - \sum_{k=-N}^{N} f_3(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right|$$

converges effectively to zero as N tends to infinity, and further, that $f_3(t) \in \mathbb{C}_c$ for all $t \in \mathbb{R}_c$. Note that this is not sufficient for f_3 being in $\mathcal{CB}_\pi^1$, as the next corollary shows.

Corollary 1. *There exists a function $f_3 \in \mathcal{B}_\pi^1$ such that*

1) $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$,
2) $f_3 \notin \mathcal{CB}_\pi^1$, i.e., f_3 cannot be effectively approximated by elementary computable functions in $\mathcal{CB}_\pi^1$.

Proof. Corollary 1 is a direct consequence of Theorem 6, because $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$ implies that $f_3 \notin \mathcal{CB}_\pi^1$ according to Remark 1. □

Remark 7. In Corollary 1 we have a function f_3 such that the restriction $f_3|_\mathbb{Z}$ is a computable discrete-time signal where we can effectively control the approximation error. However, the signal f_3 can never be effectively approximated by elementary computable functions in $\mathcal{CB}_\pi^1$.

Now we give the proof of Theorem 6. The main idea of the proof is to employ the fact that even if for a signal in $\mathcal{B}_\pi^1$ we can effectively control the ℓ^1-norm, i.e., the discrete-time behavior, it is generally not possible to effectively control the continuous-time behavior.

Proof of Theorem 6. For $k \in \mathbb{N}$, $k \geq 1$, let

$$\begin{aligned} g_k(t) &= \frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi(t+2k))}{\pi(t+2k)} \\ &= \frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi t)}{\pi(t+2k)} \\ &= \frac{2k\sin(\pi t)}{\pi t(t+2k)}, \qquad t \in \mathbb{R}. \end{aligned}$$

Thus, we have $g_k \in \mathcal{CB}_\pi^1$ and $\int_{-\infty}^{\infty} |g_k(t)|\ \mathrm{d}t \in \mathbb{R}_c$ for all $k \in \mathbb{N}$. For $N \in \mathbb{N}$, we consider

$$q_N(t) = \frac{1}{N}\sum_{k=1}^{N} g_k(t), \qquad t \in \mathbb{R}.$$

We have $q_N \in \mathcal{B}_\pi^1$ for all $N \in \mathbb{N}$. Moreover, since

$$q_N(t) = \frac{\sin(\pi t)}{\pi t} - \frac{1}{N}\sum_{k=1}^{N} \frac{\sin(\pi(t+2k))}{\pi(t+2k)},$$

we see that q_N is an elementary computable function, and as a consequence, we have $q_N \in \mathcal{CB}_\pi^1$ as well as $\|q_N\|_{\mathcal{B}_\pi^1} \in \mathbb{R}_c$ for all $N \in \mathbb{N}$. Hence $\{q_N\}_{N\in\mathbb{N}}$ is a computable sequence of functions in $\mathcal{CB}_\pi^1$. Further, we have $q_N(0) = 1$, $q_N(k) = 0$ for all $|k| > 2N$, $q_N(2l) = -1/N$ for all $l = -1, -2, \ldots, -N$, as well as $q_N(l) = 0$ for all remaining arguments l. It follows that

$$\sum_{l=-\infty}^{\infty} |q_N(l)| = 1 + \sum_{l=-N}^{-1} \frac{1}{N} = 2. \tag{7}$$

We consider the computable function

$$a(t) = \begin{cases} 0, & t < 0, \\ t\sin(\pi t), & 0 \leq t < 1, \\ \sin(\pi t), & t \geq 1, \end{cases} \qquad t \in \mathbb{R},$$

and set

$$C_N^{(1)} = \int_0^{\infty} q_N(t)a(t)\ \mathrm{d}t.$$

For $M \in \mathbb{N}$, we have

$$\begin{aligned} \left| C_N^{(1)} - \int_0^M q_N(t)a(t)\ \mathrm{d}t \right| &= \left| \int_M^{\infty} q_N(t)a(t)\ \mathrm{d}t \right| \\ &\leq \int_M^{\infty} \frac{1}{N}\sum_{k=1}^{N} |g_k(t)|\ \mathrm{d}t = \frac{1}{N}\sum_{k=1}^{N}\int_M^{\infty} |g_k(t)|\ \mathrm{d}t. \end{aligned}$$

Since $g_k \in \mathcal{CB}_\pi^\infty$, it follows that there exists a recursive function ξ_k such that for all $R \in \mathbb{N}$ we have

$$\int_{M_k}^{\infty} |g_k(t)|\ \mathrm{d}t < \frac{1}{2^R}$$

for all $M_k \geq \xi_k(R)$. For $N \in \mathbb{N}$ and $R \in \mathbb{N}$, we set

$$\overline{M}(N,R) = \max_{k=1,\ldots,N} \xi_k(R).$$

Note that $\overline{M}$ is a recursive function. It follows that

$$\begin{aligned} \left| C_N^{(1)} - \int_0^M q_N(t)a(t)\ \mathrm{d}t \right| &\leq \frac{1}{N}\sum_{k=1}^{N}\int_{\xi_k(R)}^{\infty} |g_k(t)|\ \mathrm{d}t \\ &< \frac{1}{2^R} \end{aligned}$$

for all $M \geq \overline{M}(N,R)$. Thus, the sequence $\{\int_0^M q_N(t)a(t)\ \mathrm{d}t\}_{M\in\mathbb{N}}$ of computable numbers converges to $C_N^{(1)}$ effectively in N and M. Hence $\{C_N^{(1)}\}_{N\in\mathbb{N}}$ is a computable sequence of computable numbers.

Let $A \subset \mathbb{N}$ be a recursively enumerable non-recursive set and $\phi_A \colon \mathbb{N} \to A$ a recursive enumeration of A. We consider the function

$$f_3(t) = \sum_{N=3}^{\infty} \frac{q_N(t)}{2^{\phi_A(N)} C_N^{(1)}}, \qquad t \in \mathbb{R}.$$

Next, we derive a lower bound for $C_N^{(1)}$. We have

$$C_N^{(1)} = \int_0^1 q_N(t)a(t)\ \mathrm{d}t + \int_1^{\infty} q_N(t)a(t)\ \mathrm{d}t. \tag{8}$$

For the second integral we have

$$\begin{aligned} &\int_1^{\infty} q_N(t)a(t)\ \mathrm{d}t \\ &= \int_1^{\infty} \left(\frac{\sin(\pi t)}{\pi t} - \frac{1}{N}\sum_{k=1}^{N} \frac{\sin(\pi(t+2k))}{\pi(t+2k)} \right) \sin(\pi t)\ \mathrm{d}t \\ &= \int_1^{\infty} \frac{(\sin(\pi t))^2}{N\pi t} \sum_{k=1}^{N} \frac{2k}{t+2k}\ \mathrm{d}t \\ &> \int_1^{N} \frac{(\sin(\pi t))^2}{N\pi t} \sum_{k=1}^{N} \frac{2k}{t+2k}\ \mathrm{d}t. \end{aligned}$$

For $t \in [1, N]$ and $1 \leq k \leq N$, we have

$$\frac{2k}{t+2k} \geq \frac{2k}{N+2k} \geq \frac{2k}{N+2N} = \frac{2k}{3N},$$

and consequently

$$\frac{1}{N}\sum_{k=1}^{N} \frac{2k}{t+2k} \geq \frac{2}{3N^2}\sum_{k=1}^{N} k = \frac{2N(N+1)}{6N^2} > \frac{N^2}{3N^2} = \frac{1}{3}.$$

Hence we obtain

$$\int_1^{\infty} q_N(t)a(t)\ \mathrm{d}t > \frac{1}{3\pi}\int_1^{N} \frac{(\sin(\pi t))^2}{t}\ \mathrm{d}t.$$

We further have

$$\begin{aligned} \int_1^{N} \frac{(\sin(\pi t))^2}{t}\ \mathrm{d}t &= \sum_{k=1}^{N-1}\int_k^{k+1} \frac{(\sin(\pi t))^2}{t}\ \mathrm{d}t \\ &> \sum_{k=1}^{N-1} \frac{1}{k+1}\int_k^{k+1} (\sin(\pi t))^2\ \mathrm{d}t = \frac{1}{2}\sum_{k=1}^{N-1}\frac{1}{k+1} \\ &> \frac{1}{2}\int_1^{N} \frac{1}{\tau+1}\ \mathrm{d}\tau > \frac{1}{2}\log\left(\frac{N}{2}\right), \end{aligned}$$

where we used that $\int_k^{k+1} (\sin(\pi t))^2\ \mathrm{d}t = 1/2$. Hence we obtain that

$$\int_1^{\infty} q_N(t)a(t)\ \mathrm{d}t > \frac{1}{6\pi}\log\left(\frac{N}{2}\right). \tag{9}$$

Further, for the first integral in (8) we have

$$\begin{aligned}\left|\int_0^1 q_N(t)a(t)\ \mathrm{d}t\right| &= \left|\int_0^1 \frac{1}{N}\sum_{k=1}^N g_k(t)a(t)\ \mathrm{d}t\right| \\ &= \int_0^1 \frac{1}{N}\sum_{k=1}^N \frac{2k(\sin(\pi t))^2}{\pi(t+2k)}\ \mathrm{d}t \le \int_0^1 \frac{1}{N}\sum_{k=1}^N \frac{2k}{\pi 2k}\ \mathrm{d}t \\ &= \frac{1}{\pi}. \end{aligned} \tag{10}$$

Combining (8), (9), and (10), we see that

$$\begin{aligned} C_N^{(1)} &\ge \int_1^\infty q_N(t)a(t)\ \mathrm{d}t - \left|\int_0^1 q_N(t)a(t)\ \mathrm{d}t\right| \\ &> \frac{1}{6\pi}\log\left(\frac{N}{2}\right) - \frac{1}{\pi}. \end{aligned} \tag{11}$$

Note that we have constructed a sequence $\{q_N\}_{N\in\mathbb{N}}$ of elementary computable functions such that for all $N \in \mathbb{N}$ we have

$$\frac{1}{6\pi}\log\left(\frac{N}{2}\right) - \frac{1}{\pi} < \int_0^\infty q_N(t)a(t)\ \mathrm{d}t = C_N^{(1)} < \infty,$$

and $|q_N(k)| \le 1$ for all $k \in \mathbb{Z}$, as well as $q_N(k) \ne 0$ for only finitely many $k \in \mathbb{Z}$.

Next, we derive an upper bound for $\|q_N\|_{\mathcal{B}_\pi^1}$. We have

$$\begin{aligned} &\int_0^\infty |q_N(t)|\ \mathrm{d}t \\ &= \int_0^\infty \frac{1}{N}\left|\sum_{k=1}^N \left(\frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi(t+2k))}{\pi(t+2k)}\right)\right|\ \mathrm{d}t \\ &= \int_0^1 \frac{\sin(\pi t)}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t \\ &\quad + \int_1^\infty \frac{|\sin(\pi t)|}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t. \end{aligned} \tag{12}$$

For the first integral in (12) we have

$$\begin{aligned} &\int_0^1 \frac{\sin(\pi t)}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t \\ &\qquad = \int_0^1 \frac{\sin(\pi t)}{N\pi}\sum_{k=1}^N \left(\frac{2k}{t(t+2k)}\right)\ \mathrm{d}t \\ &\qquad < \int_0^1 \frac{\sin(\pi t)}{\pi t}\ \mathrm{d}t < 1. \end{aligned} \tag{13}$$

For $M \ge 1$, we have

$$\begin{aligned} &\int_1^M \frac{|\sin(\pi t)|}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t \\ &< \int_1^M \frac{1}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t \\ &= \frac{1}{N}\sum_{k=1}^N \frac{1}{\pi}\left(\int_1^M \frac{1}{t}\ \mathrm{d}t - \int_1^M \frac{1}{t+2k}\ \mathrm{d}t\right) \\ &= \frac{1}{N\pi}\sum_{k=1}^N \log\left(\frac{M(1+2k)}{M+2k}\right), \end{aligned}$$

and, as a consequence, for the second integral in (12),

$$\begin{aligned} &\int_1^\infty \frac{|\sin(\pi t)|}{N}\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\ \mathrm{d}t \\ &\le \lim_{M\to\infty} \frac{1}{N\pi}\sum_{k=1}^N \log\left(\frac{M(1+2k)}{M+2k}\right) \\ &= \frac{1}{N\pi}\sum_{k=1}^N \log(1+2k) < \frac{1}{\pi}\log(2N+1). \end{aligned} \tag{14}$$

Combining (12), (13), and (14), we see that

$$\int_0^\infty |q_N(t)|\ \mathrm{d}t < 1 + \frac{1}{\pi}\log(2N+1). \tag{15}$$

Next, we treat the integral covering the negative reals. We have

$$\begin{aligned} &\int_{-\infty}^{-2N-1} |q_N(t)|\ \mathrm{d}t \\ &= \int_{-\infty}^{-2N-1} |\sin(\pi t)|\frac{1}{N}\left|\sum_{k=1}^N \left(\frac{1}{\pi t} - \frac{1}{\pi(t+2k)}\right)\right|\ \mathrm{d}t \\ &= \int_{2N+1}^\infty |\sin(\pi t)|\frac{1}{N}\sum_{k=1}^N \left(\frac{1}{\pi(t-2k)} - \frac{1}{\pi t}\right)\ \mathrm{d}t. \end{aligned}$$

For $M > 2N+1$ we have

$$\begin{aligned} &\int_{2N+1}^M |\sin(\pi t)|\frac{1}{N}\sum_{k=1}^N \left(\frac{1}{\pi(t-2k)} - \frac{1}{\pi t}\right)\ \mathrm{d}t \\ &< \int_{2N+1}^M \frac{1}{N\pi}\sum_{k=1}^N \left(\frac{1}{t-2k} - \frac{1}{t}\right)\ \mathrm{d}t \\ &= \frac{1}{N\pi}\sum_{k=1}^N \left(\int_{2N+1}^M \frac{1}{t-2k}\ \mathrm{d}t - \int_{2N+1}^M \frac{1}{t}\ \mathrm{d}t\right) \\ &= \frac{1}{N\pi}\sum_{k=1}^N \log\left(\frac{(M-2k)(2N+1)}{(2N+1-2k)M}\right). \end{aligned}$$

Hence we see that

$$\begin{aligned} &\int_{2N+1}^\infty \frac{|\sin(\pi t)|}{N}\sum_{k=1}^N \left(\frac{1}{\pi(t-2k)} - \frac{1}{\pi t}\right)\ \mathrm{d}t \\ &\le \lim_{M\to\infty} \frac{1}{N\pi}\sum_{k=1}^N \log\left(\frac{(M-2k)(2N+1)}{(2N+1-2k)M}\right) \\ &= \frac{1}{N\pi}\sum_{k=1}^N \log\left(\frac{2N+1}{2N+1-2k}\right) < \frac{1}{\pi}\log(2N+1), \end{aligned}$$

which implies that

$$\int_{-\infty}^{-2N-1} |q_N(t)|\ \mathrm{d}t < \frac{1}{\pi}\log(2N+1). \tag{16}$$

Further, we have

$$\begin{aligned}
&\int_{-2N-1}^{0} |q_N(t)|\ \mathrm{d}t \\
&= \int_{-2N-1}^{0} \frac{1}{N} \left| \sum_{k=1}^{N} \left(\frac{\sin(\pi t)}{\pi t} - \frac{\sin(\pi(t+2k))}{\pi(t+2k)} \right) \right| \ \mathrm{d}t \\
&\leq \int_{-2N-1}^{0} \frac{1}{N} \sum_{k=1}^{N} \left(\left| \frac{\sin(\pi t)}{\pi t} \right| + \left| \frac{\sin(\pi(t+2k))}{\pi(t+2k)} \right| \right) \ \mathrm{d}t \\
&= \int_{0}^{2N+1} \left| \frac{\sin(\pi t)}{\pi t} \right| \ \mathrm{d}t + \frac{1}{N} \sum_{k=1}^{N} \int_{0}^{2N+1} \left| \frac{\sin(\pi(t-2k))}{\pi(t-2k)} \right| \ \mathrm{d}t.
\end{aligned}$$

For the first integral we obtain

$$\begin{aligned}
\int_{0}^{2N+1} \left| \frac{\sin(\pi t)}{\pi t} \right| \ \mathrm{d}t &< 1 + \int_{1}^{2N+1} \left| \frac{\sin(\pi t)}{\pi t} \right| \ \mathrm{d}t \\
&< 1 + \frac{1}{\pi} \int_{1}^{2N+1} \frac{1}{t} \ \mathrm{d}t = 1 + \frac{1}{\pi} \log(2N+1).
\end{aligned}$$

For the second term we obtain

$$\begin{aligned}
&\frac{1}{N} \sum_{k=1}^{N} \int_{0}^{2N+1} \left| \frac{\sin(\pi(t-2k))}{\pi(t-2k)} \right| \ \mathrm{d}t \\
&\quad = \frac{1}{N} \sum_{k=1}^{N} \int_{-2k}^{2N+1-2k} \left| \frac{\sin(\pi t)}{\pi t} \right| \ \mathrm{d}t \\
&\quad < \frac{1}{N} \sum_{k=1}^{N} \int_{-2N-1}^{2N+1} \left| \frac{\sin(\pi t)}{\pi t} \right| \ \mathrm{d}t \\
&\quad < 2 \left(1 + \frac{1}{\pi} \int_{1}^{2N+1} \frac{1}{t} \ \mathrm{d}t \right) \\
&\quad = 2 + \frac{2}{\pi} \log(2N+1).
\end{aligned}$$

It follows that

$$\int_{-2N-1}^{0} |q_N(t)|\ \mathrm{d}t < 3 + \frac{3}{\pi} \log(2N+1). \tag{17}$$

Combining (15), (16), and (17), it follows that

$$\begin{aligned}
\int_{-\infty}^{\infty} |q_N(t)|\ \mathrm{d}t &< 3 + \frac{4}{\pi} \log(2N+1) + 1 + \frac{1}{\pi} \log(2N+1) \\
&= 4 + \frac{5}{\pi} \log(2N+1).
\end{aligned} \tag{18}$$

For $N \geq 3$, we have

$$\begin{aligned}
\frac{\|q_N\|_{\mathcal{B}_\pi^1}}{C_N^{(1)}} &< \frac{4}{\frac{2}{3\pi} \log\left(\frac{N}{2}\right)} + \frac{\frac{5}{\pi} \log(2N+1)}{\frac{2}{3\pi} \log\left(\frac{N}{2}\right)} \\
&= \frac{6}{\pi \log\left(\frac{N}{2}\right)} + \frac{15 \log(2N+1)}{2 \log\left(\frac{N}{2}\right)}.
\end{aligned} \tag{19}$$

Since

$$\left(\frac{\log(2x+1)}{\log\left(\frac{x}{2}\right)} \right)' = \frac{\frac{1}{2x+1} \log\left(\frac{x}{2}\right) - \frac{2}{x} \log(2x+1)}{\left(\log\left(\frac{x}{2}\right)\right)^2} < 0$$

for $x \geq 3$, we see that the right-hand side of (19) is monotonically decreasing in N. Hence we have

$$\frac{\|q_N\|_{\mathcal{B}_\pi^1}}{C_N^{(1)}} < \frac{6}{\pi \log\left(\frac{3}{2}\right)} + \frac{15 \log(6+1)}{2 \log\left(\frac{3}{2}\right)} =: C_1.$$

Thus, it follows that

$$\begin{aligned}
\|f_3\|_{\mathcal{B}_\pi^1} &\leq \sum_{N=3}^{\infty} \frac{\|q_N\|_{\mathcal{B}_\pi^1}}{2^{\phi_A(N)} C_N^{(1)}} < C_1 \sum_{N=3}^{\infty} \frac{1}{2^{\phi_A(N)}} \\
&\leq C_1 \sum_{N=1}^{\infty} \frac{1}{2^N} = C_1,
\end{aligned} \tag{20}$$

where we used (2) in the third inequality. This shows that $f_3 \in \mathcal{B}_\pi^1$.

Next, we show that $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$. We do a proof by contradiction. Assume that $\|f_3\|_{\mathcal{B}_\pi^1} \in \mathbb{R}_c$. Since the sequence $\{\int_{|t|<M} |f_3(t)|\ \mathrm{d}t\}_{M\in\mathbb{N}}$ is monotonically increasing and converges to $\|f_3\|_{\mathcal{B}_\pi^1}$, it follows from Lemma 2 that $\{\int_{|t|\geq M} |f_3(t)|\ \mathrm{d}t\}_{M\in\mathbb{N}}$ converges effectively to zero. As a consequence, $\{\int_M^\infty |f_3(t)|\ \mathrm{d}t\}_{M\in\mathbb{N}}$ converges effectively to zero. We have

$$\begin{aligned}
\int_0^\infty f_3(t) a(t)\ \mathrm{d}t &= \sum_{N=3}^{\infty} \frac{1}{2^{\phi_A(N)} C_N^{(1)}} \int_0^\infty q_N(t) a(t)\ \mathrm{d}t \\
&= \sum_{N=3}^{\infty} \frac{1}{2^{\phi_A(N)}} \notin \mathbb{R}_c,
\end{aligned}$$

where the exchange of integration and summation is justified according to Fubibi's theorem, because

$$\sum_{N=3}^{\infty} \int_0^\infty \left| \frac{q_N(t)}{2^{\phi_A(N)} C_N^{(1)}} a(t) \right| \ \mathrm{d}t \leq \sum_{N=3}^{\infty} \frac{\|q_N\|_{\mathcal{B}_\pi^1}}{2^{\phi_A(N)} C_N^{(1)}} < C_1,$$

using the same calculation as in (20).

We have $f_3 \in \mathcal{CB}_\pi^2$ because $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$. Hence for all $M \in \mathbb{N}$ the function $f_3(t)a(t)$, $t \in [0, M]$, is a computable continuous function. According to [25, p. 37, Corollary 6a] we have $\int_0^M f_3(t)a(t)\ \mathrm{d}t \in \mathbb{R}_c$. Hence $\{\int_0^M f_3(t)a(t)\ \mathrm{d}t\}_{M\in\mathbb{N}}$ is a computable sequence of computable numbers. Since

$$\left| \int_0^\infty f_3(t)a(t)\ \mathrm{d}t - \int_0^M f_3(t)a(t)\ \mathrm{d}t \right| \leq \int_M^\infty |f_3(t)|\ \mathrm{d}t,$$

and since $\{\int_M^\infty |f_3(t)|\ \mathrm{d}t\}_{M\in\mathbb{N}}$ converges effectively to zero, it follows that the computable sequence of computable numbers $\{\int_0^M f_3(t)a(t)\ \mathrm{d}t\}_{M\in\mathbb{N}}$ converges effectively to

$$\int_0^\infty f_3(t)a(t)\ \mathrm{d}t. \tag{21}$$

This implies that (21) is a computable number. This is a contradiction. Therefore, we have $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$.

Next, we prove that $f_3|_\mathbb{Z} \in \mathcal{C}\ell^1$, i.e., the first statement of the theorem. We have

$$\begin{aligned}
&\left\| f_3|_\mathbb{Z} - \sum_{N=3}^{M} \frac{q_N|_\mathbb{Z}}{2^{\phi_A(N)} C_N^{(1)}} \right\|_{\ell^1} \leq \sum_{N=M+1}^{\infty} \frac{\|q_N|_\mathbb{Z}\|_{\ell^1}}{2^{\phi_A(N)} C_N^{(1)}} \\
&< 2 \left(\frac{1}{6\pi} \log\left(\frac{N}{2}\right) - \frac{1}{\pi} \right)^{-1} \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_A(N)}} \\
&< \left(\frac{1}{3\pi} \log\left(\frac{N}{2}\right) - \frac{2}{\pi} \right)^{-1} \sum_{N=1}^{\infty} \frac{1}{2^N} \\
&= \left(\frac{1}{3\pi} \log\left(\frac{N}{2}\right) - \frac{2}{\pi} \right)^{-1},
\end{aligned}$$

where we used (7) and (11) in the second inequality. Hence we see that the computable sequence

$$\left\{ \sum_{N=3}^{M} \frac{q_N|_{\mathbb{Z}}}{2^{\phi_A(N)} C_N^{(1)}} \right\}_{M=3}^{\infty}$$

converges effectively to $f_3|_{\mathbb{Z}}$ in the ℓ^1-norm, implying that $f_3|_{\mathbb{Z}} \in \mathcal{C}\ell^1$. □

VI. Connections of Computability for $\mathcal{B}_\pi^2$

In this section we will analyze the question whether the two conditions: 1) $f \in \mathcal{B}_\pi^2$ and 2) f is a computable continuous function are sufficient for $f \in \mathcal{CB}_\pi^2$. The next theorem shows that this question has to be answered in the negative.

Theorem 7. *There exists a function $f_4 \in \mathcal{B}_\pi^2$ such that*

1) *$f_4(t) \in \mathbb{R}_c$ for all $t \in \mathbb{R}_c$,*
2) *there exists a recursive function $\xi\colon \mathbb{N}^2 \to \mathbb{N}$ such that for all $T \in \mathbb{N}$ and $M \in \mathbb{N}$ we have for all $N \geq \xi(T, N)$*
$$\max_{t \in [-T,T]} \left| f_4(t) - \sum_{k=-N}^{N} f_4(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| < \frac{1}{2^M},$$
3) *the sequence*
$$\left\{ \max_{t \in \mathbb{R}} \left| f_4(t) - \sum_{k=-N}^{N} f_4(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| \right\}_{N \in \mathbb{N}} \qquad (22)$$
does not converge effectively to zero, and
4) *$\|f_4\|_{\mathcal{B}_\pi^2} \notin \mathbb{R}_c$ as well as $f_4 \notin \mathcal{CB}_\pi^2$.*

Remark 8. Item 2) implies that f_4 is a computable continuous function on $\mathbb{R}$.

Proof. Let $A \subset \mathbb{N}$ be a recursively enumerable non-recursive set and $\phi_A\colon \mathbb{N} \to A$ a recursive enumeration of A. We consider the function

$$f_4(t) = \sum_{n=1}^{\infty} \frac{1}{2^{\phi_A(n)}} \frac{\sin(\pi(t-n))}{\pi(t-n)}, \quad t \in \mathbb{R}. \qquad (23)$$

We have

$$\|f_4\|_{\mathcal{B}_\pi^2} = \left(\sum_{n=1}^{\infty} \frac{1}{2^{2\phi_A(n)}} \right)^{\frac{1}{2}} < \infty,$$

i.e., $f_4 \in \mathcal{B}_\pi^2$. Further, the series in (23) converges globally uniformly. Let $T \in \mathbb{N}$ and $t \in [-T, T]$ be arbitrary. For $N > T$, we have

$$\begin{aligned} &\left| f_4(t) - \sum_{n=1}^{N} \frac{1}{2^{\phi_A(n)}} \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| \\ &\leq \sum_{n=N+1}^{\infty} \frac{1}{2^{\phi_A(n)}} \left| \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| \leq \sum_{n=N+1}^{\infty} \frac{1}{2^{\phi_A(n)} \pi |t-n|} \\ &\leq \frac{1}{N+1-T} \sum_{n=N+1}^{\infty} \frac{1}{2^{\phi_A(n)}} < \frac{1}{N+1-T}, \end{aligned} \qquad (24)$$

where we used (2) in the last inequality. For $T \in \mathbb{N}$ and $M \in \mathbb{N}$, we set $\xi(T, M) = 2^M + T - 1$. Clearly ξ is a recursive function, and because of (24), we have for all $N \geq \xi(T, M)$ that

$$\left| f_4(t) - \sum_{n=1}^{N} \frac{1}{2^{\phi_A(n)}} \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| < \frac{1}{2^M}.$$

This proves item 2) of the theorem.

For $t \in \mathbb{R}_c$, the sequence

$$\left\{ \sum_{n=1}^{M} f_4(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \right\}_{M \in \mathbb{N}}$$

is a computable sequence of computable numbers that converges effectively to $f_4(t)$, according to item 2) of the theorem. Hence we have $f_4(t) \in \mathbb{R}_c$ for all $t \in \mathbb{R}_c$. This proves item 1) of the theorem.

Next, we prove item 3) of the theorem. We do a proof by contradiction and assume that (22) converges effectively to zero. That is, we assume that there exists a recursive function $\xi_4\colon \mathbb{N} \to \mathbb{N}$ such that for all $M \in \mathbb{N}$ and $N \geq \xi_4(M)$ we have

$$\max_{t \in \mathbb{R}} \left| f_4(t) - \sum_{n=1}^{N} f_4(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| < \frac{1}{2^M}.$$

Let $\eta \in \mathbb{N}$ and $M > \eta$ be arbitrary. For all $t \in \mathbb{R}$ and all $N \geq \xi_4(M)$ we have

$$\begin{aligned} &\left| \sum_{n=N+1}^{\infty} f_4(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| \\ &\qquad = \left| f_4(t) - \sum_{n=1}^{N} f_4(n) \frac{\sin(\pi(t-n))}{\pi(t-n)} \right| < \frac{1}{2^M}. \end{aligned}$$

Thus, for $n \geq N+1$ it follows that

$$\frac{1}{2^{\phi_A(n)}} = f_4(n) = \left| \sum_{k=N+1}^{\infty} f_4(k) \frac{\sin(\pi(n-k))}{\pi(n-k)} \right| < \frac{1}{2^M},$$

which shows that $\phi_A(n) > M$ for all $n \geq N+1$. We compute the list of numbers $\{\phi_A(1), \phi_A(2), \dots, \phi_A(N)\}$ and check if η is in this list. If η is in this list then we have $\eta \in A$. If η is not in this list then we have $\eta \in A^{\complement}$, because we know that $\phi_A(n) > M > \eta$ for all $n \geq N+1$. This shows that A is a recursive set, because the above procedure gives us an algorithm that can decide for arbitrary $\eta \in \mathbb{N}$ whether $\eta \in A$ or $\eta \in A^{\complement}$. This is a contradiction. Hence (22) does not converge effectively to zero.

Finally, we prove item 4) of the theorem. We do a proof by contradiction and assume that $\|f_4\|_{\mathcal{B}_\pi^2} \in \mathbb{R}_c$. Since $\{\sum_{k=-N}^{N} |f_4(k)|^2\}_{N \in \mathbb{Z}}$ is a monotonically increasing sequence of computable numbers that converges to $\|f_4\|_{\mathcal{B}_\pi^2} \in \mathbb{R}_c$, it follows from Lemma 2 that $\{\sum_{|k|>N} |f_4(k)|^2\}_{N \in \mathbb{Z}}$ convergences effectively to zero. Further, since

$$\|f_4 - S_N f_4\|_{\mathcal{B}_\pi^2}^2 = \sum_{|k|>N} |f_4(k)|^2,$$

we see that $\{S_N f_4\}_{N \in \mathbb{N}}$ converges effectively to f_4 in the $\mathcal{B}_\pi^2$-norm. Since $\|f_4\|_\infty \leq \|f_4\|_{\mathcal{B}_\pi^2}$, this implies that (22) converges effectively to zero. This is a contradiction. Hence it follows that $\|f_4\|_{\mathcal{B}_\pi^2} \notin \mathbb{R}_c$. Further, $\|f_4\|_{\mathcal{B}_\pi^2} \notin \mathbb{R}_c$ implies that $f_4 \notin \mathcal{CB}_\pi^2$, according to Remark 1. □

VII. Applications

A. BIBO Stability

We set $h = f_3$, where $f_3 \in \mathcal{B}_\pi^1$ is the function from Theorem 6, and consider the discrete-time linear time-invariant (LTI) system

$$(T_1^{\mathrm{d}}x)(k) = \sum_{l=-\infty}^{\infty} h(k-l)x(l) = \sum_{l=-\infty}^{\infty} h(l)x(k-l)$$

for input signals $x \in \ell^\infty$. This system is BIBO stable and the BIBO-norm is given by

$$\|T_1^{\mathrm{d}}\|_{\mathrm{BIBO}}^{\mathrm{d}} = \sup_{\|x\|_{\ell^\infty} \le 1} \|T_1^{\mathrm{d}}x\|_{\ell^\infty} = \sum_{l=-\infty}^{\infty} |h(k)|.$$

Note that we have

$$\sum_{l=-\infty}^{\infty} |h(k)| \in \mathbb{R}_c.$$

Hence the BIBO-norm $\|T_1^{\mathrm{d}}\|_{\mathrm{BIBO}}^{\mathrm{d}}$ is a computable number, and we can approximate this number with an effective control of the approximation error.

Next, we consider the continuous-time system

$$(T_1^{\mathrm{c}}x)(k) = \int_{-\infty}^{\infty} h(t-\tau)f(\tau)\,\mathrm{d}\tau = \int_{-\infty}^{\infty} h(\tau)f(t-\tau)\,\mathrm{d}\tau$$

for input signals $f \in L^\infty(\mathbb{R})$. Both integrals converge, because $h \in \mathcal{B}_\pi^1$. Further, we have for the continuous-time BIBO-norm

$$\|T_1^{\mathrm{c}}\|_{\mathrm{BIBO}}^{\mathrm{c}} = \sup_{\|f\|_\infty \le 1} \|T_1^{\mathrm{c}}f\|_\infty = \int_{-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau.$$

Here we are in the situation that

$$\int_{-\infty}^{\infty} |h(\tau)|\,\mathrm{d}\tau \notin \mathbb{R}_c.$$

That is, we cannot algorithmically determine the continuous-time BIBO-norm $\|T_1^{\mathrm{c}}\|_{\mathrm{BIBO}}^{\mathrm{c}}$ on a digital computer, even though that is possible for the corresponding discrete-time system and even though h is bandlimited with bandwidth π.

B. Concentration in the Time-Domain

Using the signal f_3 from Theorem 6, we can gain some insights into the time-domain concentration behavior of bandlimited signals and its algorithmic characterization.

Bandlimited signals posses a perfect concentration in the frequency domain in the sense that their Fourier transforms are non-zero only on some finite interval. Because of the perfect concentration in the frequency domain, they cannot simultaneously be—with the exception of the zero signal—perfectly concentrated in the time-domain.

For a signal $f \in \mathcal{B}_\pi^1$, the expression

$$\int_{-L}^{L} |f(t)|\,\mathrm{d}t \tag{25}$$

can be considered as a measure of the "amount" of the signal f that is located within the interval $[-L, L]$. Further, the expression

$$\int_{-\infty}^{\infty} |f(t)|\,\mathrm{d}t - \int_{-L}^{L} |f(t)|\,\mathrm{d}t = \int_{|t|>L} |f(t)|\,\mathrm{d}t \tag{26}$$

can be seen as a measure of the concentration of the continuous-time signal f on the time interval $[-L, L]$. The smaller the value, the more concentrated the signal is on the interval. Hence the study of the time concentration behavior is closely related to the question how fast the sequence of functions $\{f_L\}_{L\in\mathbb{N}}$, given by

$$f_L(t) = \begin{cases} f(t), & |t| \le L, \\ 0, & |t| > L, \end{cases}$$

converges to f in the $L^1(\mathbb{R})$-norm. For a discrete-time signal, the time concentration is described by the analogous expressions, where the integrals are replaced with sums.

Note that even though f is bandlimited, the functions f_L, $L \in \mathbb{N}$, are no longer bandlimited, except for the trivial case where $f \equiv 0$. Hence the signals f_L, $L \in \mathbb{N}$, cannot be analyzed using the same techniques that were employed to study the computability of bandlimited signals. Nevertheless, the results in this paper allow us to make a statement about the time concentration.

For all signals $f \in \mathcal{B}_\pi^1$, the expression in (26) converges to zero as L tends to infinity. The question now is whether, and under what conditions on f, this convergence is effective, i.e., can be algorithmically described. If $f \in \mathcal{CB}_\pi^1$, then there exists a computable sequence $\{f_N\}_{N\in\mathbb{N}}$ of elementary computable functions such that

$$\|f - f_N\|_{\mathcal{B}_\pi^1} \le \frac{1}{2^N},$$

and it follows that

$$\left| \|f\|_{\mathcal{B}_\pi^1} - \|f_N\|_{\mathcal{B}_\pi^1} \right| \le \frac{1}{2^N},$$

which shows that $\|f\|_{\mathcal{B}_\pi^1} \in \mathbb{R}_c$. Further, we have

$$\begin{aligned} \left| \sum_{k=-\infty}^{\infty} |f(k)| - \sum_{k=-\infty}^{\infty} |f_N(k)| \right| &\le \sum_{k=-\infty}^{\infty} |f(k) - f_N(k)| \\ &\le \frac{1}{A_1} \|f - f_N\|_{\mathcal{B}_\pi^1} \le \frac{1}{A_1 2^N}, \end{aligned}$$

where we used Lemma 1 in the second inequality, which implies that $\|f|_{\mathbb{Z}}\|_{\ell^1} \in \mathbb{R}_c$. As a consequence of Lemma 2, the monotonically increasing sequences $\{\int_{-L}^{L} |f(t)|\,\mathrm{d}t\}_{L\in\mathbb{N}}$ and $\{\sum_{k=-L}^{L} |f(k)|\}_{L\in\mathbb{N}}$ converge effectively to $\|f\|_{\mathcal{B}_\pi^1}$ and $\|f|_{\mathbb{Z}}\|_{\ell^1}$, respectively. Therefore, we have an algorithmic description of the time-domain concentration behavior.

In general, this result does not hold for $\mathcal{B}_\pi^1$. In Theorem 6 we had the signal $f_3 \in \mathcal{B}_\pi^1$ with $\|f_3|_{\mathbb{Z}}\|_{\ell^1} \in \mathbb{R}_c$ but $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$. Hence the concentration of the discrete-time signal $f_3|_{\mathbb{Z}}$ can be algorithmically described, whereas this is not possible for the continuous-time signal f_3, because $\|f_3\|_{\mathcal{B}_\pi^1} \notin \mathbb{R}_c$. Thus, for f_3 and the corresponding discrete-time signal $f_3|_{\mathbb{Z}}$, we see a strong difference regarding the feasibility of describing the time-domain concentration algorithmically.

Remark 9. In the example above, i.e., for $\mathcal{CB}_\pi^1$, the time-domain concentration behavior of f_N also gives information of the time-domain concentration behavior of f. It would be interesting to have results for $f \in \mathcal{CB}_\pi^p$, $1 < p < \infty$, that connect the time concentration behavior, i.e., the effective convergence of $(\int_{-L}^{L} |f(t)|^p\,\mathrm{d}t)^{1/p}$ to $\|f\|_{\mathcal{B}_\pi^p}$, with the effective approximation of f in the L^p-norm by finite Shannon sampling series. For that purpose, however, new approaches have to be developed, because the involved functions are no longer bandlimited.

VIII. Conclusion

In this paper we treated bandlimited signals and characterized the range of signal spaces for which the Shannon sampling series provides an effective approximation process. We also showed that there are computable discrete-time signals in $\mathcal{C}\ell^1$ for which the corresponding analog signal cannot be computed on a digital computer, because we cannot effectively control the approximation error.

Recent studies have also shown for other signal processing operations, such as downsampling and the Fourier transform [34], [36], [44], that even though the required limit processes occurring in those operations converge classically, they might not converge effectively. As a consequence, those operations cannot be implemented on a digital computer.

The results are also of fundamental importance for the reemerging field of analog computing. One of the key advantages of digital computers compared to analog computers is their robustness. However, the prevalent conception that an ideal digital computer, represented by a Turing machine, can in principle solve the same class of problems as an ideal analog computer, is not correct. For example, the Fourier transform is not always computable on a digital computer, whereas the ideal analog machine, represented by a Fourier optics setup, is capable of doing so [34]. Whether and how this theoretical superiority of the analog machine can be translated into practice is unclear and a topic for further research.

Further interesting questions and problems remain open and need to be analyzed and solved in future research. It would be interesting to conduct the analyses that were done in this paper for other signal spaces, and to find suitable building blocks similar to the elementary functions in the present paper that can be used for the effective approximation. For example, time-limited signals and causal signals, which are non-zero only on the positive time axis, could be studied. But even for bandlimited signals many questions are open: 1) What are the results for the space $\mathcal{B}^\infty_{\pi,0}$, which has important applications, e.g., in communication systems [39]? 2) What time-domain concentration behavior do we have for other signal spaces than $\mathcal{B}^1_\pi$? 3) What is the situation if oversampling or non-equidistant sampling is used? 4) What is the influence of noise?

Appendix A
Plancherel–Pólya inequality and the Shannon Sampling series

The Plancherel–Pólya inequality can be used to prove the convergence of the Shannon sampling series. Let $p \in (1,\infty)$ and $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}} \in \ell^p$. Then the series in

$$f_\alpha(t) = \sum_{k=-\infty}^{\infty} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}, \tag{27}$$

converges absolutely and globally uniformly, as well as in the L^p-norm. This can be easily seen: For $N_1 > N_2$ we have

$$\left(\int_{-\infty}^{\infty} \left| \sum_{k=-N_1}^{N_1} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)} - \sum_{k=-N_2}^{N_2} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right|^p \mathrm{d}t \right)^{\frac{1}{p}} \le B_p \left(\sum_{N_2<|k|\le N_1} |\alpha_k|^p \right)^{\frac{1}{p}},$$

where we used Lemma 1 in the last inequality. Since $\alpha \in \ell^p$, it follows that

$$\left\{ \sum_{k=-N}^{N} \alpha_k \frac{\sin(\pi(\,\cdot\,-k))}{\pi(\,\cdot\,-k)} \right\}_{N\in\mathbb{N}}$$

is a Cauchy sequence in $L^p(\mathbb{R})$. Hence the sequence in (27) converges in the L^p-norm, and we have $f_\alpha \in \mathcal{B}^p_\pi$. Further, since $\|f\|_\infty \le (1+\pi)\|f\|_{\mathcal{B}^p_\pi}$ for all $f \in \mathcal{B}^p_\pi$, we also see that the sequence in (27) converges uniformly on the real axis.

Appendix B
Approximation in $\mathcal{B}^1_\pi$ Using a Linear Combination of sinc-Functions

Next, we will show that every function $f \in \mathcal{B}^1_\pi$ can be approximated arbitrarily well in the $\mathcal{B}^1_\pi$-norm by using a finite linear combination of sinc-functions, as used in the Shannon sampling series. However, as we have seen, this approximation is not effective in general. It is clear that we cannot use the Shannon sampling series directly, because for $f \in \mathcal{B}^1_\pi$, the finite Shannon sampling series

$$(S_N f)(t) = \sum_{k=-N}^{N} f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

is not in $\mathcal{B}^1_\pi$ in general, which is due to the fact that $\sin(\pi\,\cdot)/(\pi\,\cdot) \notin \mathcal{B}^1_\pi$.

Let $f \in \mathcal{B}^1_\pi$ and $\epsilon > 0$ be arbitrary but fixed. For $0 < \delta \le 1/2$ we set $F_\delta(t) = f((1-\delta)t)$, $t \in \mathbb{R}$. For $0 < \delta \le 1/2$ and $T > 0$ we have

$$\int_{|t|\ge T} |F_\delta(t)|\,\mathrm{d}t = \frac{1}{1-\delta} \int_{|t|\ge(1-\delta)T} |f(t)|\,\mathrm{d}t \le 2 \int_{|t|\ge T/2} |f(t)|\,\mathrm{d}t.$$

Since $f \in \mathcal{B}^1_\pi$, there exists a $T_1 \in \mathbb{R}$ such that

$$2 \int_{|t|\ge T_1/2} |f(t)|\,\mathrm{d}t < \frac{\epsilon}{8}. \tag{28}$$

Hence we have

$$\int_{|t|\ge T_1} |F_\delta(t)|\,\mathrm{d}t < \frac{\epsilon}{8} \tag{29}$$

for all $0 < \delta \le 1/2$. We further have

$$\int_{-\infty}^{\infty} |f(t) - F_\delta(t)|\,\mathrm{d}t = \int_{|t|<T_1} |f(t) - F_\delta(t)|\,\mathrm{d}t + \int_{|t|\ge T_1} |f(t) - F_\delta(t)|\,\mathrm{d}t.$$

For the second integral we obtain

$$\int_{|t|\geq T_1} |f(t) - F_\delta(t)| \, \mathrm{d}t \leq \int_{|t|\geq T_1} |f(t)| \, \mathrm{d}t + \int_{|t|\geq T_1} |F_\delta(t)| \, \mathrm{d}t < \frac{\epsilon}{8} + \frac{\epsilon}{8} = \frac{\epsilon}{4},$$

where we used (28) and (29) in the second inequality. This holds for all $0 < \delta \leq 1/2$. Next, we choose $0 < \delta_0 \leq 1/2$ small enough, such that

$$\int_{|t|<T_1} |f(t) - F_{\delta_0}(t)| \, \mathrm{d}t < \frac{\epsilon}{4}.$$

Thus, we see that

$$\int_{-\infty}^{\infty} |f(t) - F_{\delta_0}(t)| \, \mathrm{d}t < \frac{\epsilon}{2}. \tag{30}$$

We have $F_{\delta_0} \in \mathcal{B}_\pi^1$ and $\hat{F}_{\delta_0}(\omega) = 0$ for $|\omega| > (1-\delta_0)\pi$. Let

$$\hat{\gamma}_{\delta_0}(\omega) = \begin{cases} 1, & |\omega| \leq (1-\delta_0)\pi, \\ \frac{\pi - |\omega|}{\delta_0 \pi}, & (1-\delta_0)\pi < |\omega| < \pi, \\ 0, & |\omega| \geq \pi. \end{cases}$$

Then we have

$$F_{\delta_0}(t) = \sum_{k=-\infty}^{\infty} F_{\delta_0}(k) \gamma_{\delta_0}(t-k),$$

because $\gamma_{\delta_0} \in \mathcal{B}_\pi^1$. Further, since $F_{\delta_0}|_{\mathbb{Z}} \in \ell^1$ according to Nikol'skiĭ's inequality [32, p. 49], we see that

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \left| F_{\delta_0}(t) - \sum_{k=-N}^{N} F_{\delta_0}(k)\gamma_{\delta_0}(t-k) \right| \, \mathrm{d}t = 0.$$

Hence there exists a natural number N_0 such that

$$\int_{-\infty}^{\infty} \left| F_{\delta_0}(t) - \sum_{k=-N_0}^{N_0} F_{\delta_0}(k)\gamma_{\delta_0}(t-k) \right| \, \mathrm{d}t < \frac{\epsilon}{4}. \tag{31}$$

Next, we will approximate γ_{δ_0} by a suitable sampling series. The derivative

$$\gamma'_{\delta_0}(\omega) = \frac{\mathrm{d}\gamma_{\delta_0}}{\mathrm{d}\omega}(\omega)$$

is a bounded piecewise linear function. For $M \in \mathbb{N}$ let

$$(\Gamma_M \gamma'_{\delta_0})(\omega) = \sum_{\substack{k=-M \\ k\neq 0}}^{M} c_k(\gamma'_{\delta_0}) \, \mathrm{e}^{i\omega k}, \quad |\omega| \leq \pi,$$

denote the M-th partial sum of the Fourier series of γ'_{δ_0}, where $c_k(\gamma'_{\delta_0})$ are the usual Fourier coefficients. Note that $c_0(\gamma'_{\delta_0}) = 0$ and $c_k(\gamma'_{\delta_0}) = -c_{-k}(\gamma'_{\delta_0})$, $k \in \mathbb{Z}$, because γ'_{δ_0} is an odd function. Further, let

$$\hat{P}_M(\omega) = \sum_{\substack{k=-M \\ k\neq 0}}^{M} c_k(\gamma'_{\delta_0}) \frac{-1}{ik} \, \mathrm{e}^{i\omega k}, \quad |\omega| \leq \pi.$$

We have $\hat{P}_M(\pi) = \hat{P}_M(-\pi)$, since $\hat{P}_M$ is an even function. We set

$$\hat{P}_M^{(1)}(\omega) = \hat{P}_M(\omega) - \hat{P}_M(\pi), \quad |\omega| \leq \pi.$$

It follows that $\hat{P}_M^{(1)}(\pi) = \hat{P}_M^{(1)}(-\pi) = 0$. We further set

$$\hat{\phi}_M(\omega) = \begin{cases} \hat{P}_M^{(1)}(\omega), & |\omega| \leq \pi, \\ 0, & |\omega| > \pi. \end{cases}$$

$\hat{\phi}_M$ is a piecewise continuously differentiable function. We have

$$\int_{-\infty}^{\infty} |t\phi_M(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\phi}'_M(\omega)|^2 \, \mathrm{d}\omega.$$

For $R > 1$ it follows that

$$\begin{aligned} \int_{1\leq|t|\leq R} |\phi_M(t)| \, \mathrm{d}t &= \int_{1\leq|t|\leq R} |\phi_M(t)| \frac{|t|}{t} \, \mathrm{d}t \\ &\leq \left(\int_{1\leq|t|\leq R} |t\phi_M(t)|^2 \, \mathrm{d}t \right)^{\frac{1}{2}} \left(\int_{1\leq|t|\leq R} \frac{1}{t^2} \, \mathrm{d}t \right)^{\frac{1}{2}} \\ &\leq \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\phi}'_M(\omega)|^2 \, \mathrm{d}\omega \right)^{\frac{1}{2}} \sqrt{2}. \end{aligned}$$

This shows that $\phi_M \in \mathcal{B}_\pi^1$. Note that due to the construction of ϕ_M, the Shannon sampling series of ϕ_M has only finitely many summands.

Since

$$\int_{-\infty}^{\infty} t^2 |\gamma_{\delta_0}(t) - \phi_M(t)|^2 \, \mathrm{d}t = \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\gamma}'_{\delta_0}(\omega) - \hat{\phi}'_M(\omega)|^2 \, \mathrm{d}\omega$$

and

$$\lim_{M\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{\gamma}'_{\delta_0}(\omega) - \hat{\phi}'_M(\omega)|^2 \, \mathrm{d}t = 0,$$

it follows that

$$\lim_{M\to\infty} \int_{-\infty}^{\infty} t^2 |\gamma_{\delta_0}(t) - \phi_M(t)|^2 \, \mathrm{d}t = 0.$$

Thus, we have

$$\begin{aligned} &\lim_{M\to\infty} \int_{|t|\geq 1} |\gamma_{\delta_0}(t) - \phi_M(t)| \, \mathrm{d}t \\ &\leq \lim_{M\to\infty} \sqrt{2} \int_{|t|\geq 1} t^2 |\gamma_{\delta_0}(t) - \phi_M(t)|^2 \, \mathrm{d}t = 0. \end{aligned}$$

We also have

$$\lim_{M\to\infty} \max_{|t|\leq 1} |\gamma_{\delta_0}(t) - \phi_M(t)| = 0.$$

Hence there exists a natural number M_0 such that

$$\int_{-\infty}^{\infty} |\gamma_{\delta_0}(t) - \phi_{M_0}(t)| \, \mathrm{d}t < \frac{\epsilon}{4(2N_0+1)} \left(\max_{|k|\leq N_0} |F_{\delta_0}(k)| \right)^{-1}. \tag{32}$$

Let

$$g(t) = \sum_{k=-N_0}^{N_0} F_{\delta_0}(k) \phi_{M_0}(t-k).$$

Since g is the finite sum of functions having a finite Shannon sampling series, it follows that g has a finite Shannon sampling series. Further, since $\phi_{M_0} \in \mathcal{B}_\pi^1$, we have $g \in \mathcal{B}_\pi^1$. Moreover, we have

$$\begin{aligned}
&\int_{-\infty}^{\infty} \left| \sum_{k=-N_0}^{N_0} F_{\delta_0}(k)\gamma_{\delta_0}(t-k) - \sum_{k=-N_0}^{N_0} F_{\delta_0}(k)\phi_{M_0}(t-k) \right| \,\mathrm{d}t \\
&\le \sum_{k=-N_0}^{N_0} |F_{\delta_0}(k)| \int_{-\infty}^{\infty} |\gamma_{\delta_0}(t-k) - \phi_{M_0}(t-k)| \,\mathrm{d}t \\
&\le (2N_0+1) \max_{|k|\le N_0} |F_{\delta_0}(k)| \int_{-\infty}^{\infty} |\gamma_{\delta_0}(t) - \phi_{M_0}(t)| \,\mathrm{d}t \\
&< \frac{\epsilon}{4}, \qquad (33)
\end{aligned}$$

where we used (32) in the last inequality.

From (30), (31), (33), and the triangle inequality, it follows that

$$\|f-g\|_{\mathcal{B}_\pi^1} = \int_{-\infty}^{\infty} \left| f(t) - \sum_{k=-N_0}^{N_0} F_{\delta_0}(k)\phi_{M_0}(t-k) \right| \,\mathrm{d}t < \epsilon.$$

Since $\epsilon > 0$ was arbitrary, the proof is complete.

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